Further Results on the Stability of Distance-Based Multi-Robot Formations

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***Abstract*— An important class of multi-robot formations is specified by desired distances between adjacent robots. In previous work, we showed that distance-based formations can be globally stabilized by negative gradient, potential field based, control laws, if and only if the formation graph is a tree. In this paper, we further examine the relation between the cycle space of the formation graph and the resulting equilibria of cyclic formations. In addition, the results are extended to the case of distance based formation control for nonholonomic agents. The results are supported through computer simulations.**

I. INTRODUCTION

Decentralized control of networked multi-agent systems is a field of increasing research interest, due to its applications in robotics and large-scale systems. A particular problem considered in the robotics' literature is that of multi-agent formation control, where agents usually represent multiple robots of similar dynamics that aim to converge to a specified pattern in the state space. The desired formation can be either static [4],[7] or moving with constant velocity [18],[20].

Two main approaches in the formation control literature can be distinguished: position-based and distance-based formation control. In the first case, agents aim to converge to desired relative position vectors with respect to a subset of the rest of the team. Control designs that guarantee position-based formation stabilization have appeared for single integrator agents [7],[15] as well as nonholonomic agents [17]. On the other hand, distance-based formations have been studied in the context of graph rigidity where a series of results have appeared in recent literature, e.g., [2], [19],[9], [13]. Roughly speaking, a formation is called rigid if the fact that all desired distances are met is sufficient for the maintenance of the distances of any pair of agents. Necessary and sufficient conditions for graph rigidity have been provided in [8], [13]. The reader is also referred to the recent PhD thesis [12] and the references therein for more information on the topic. A common factor in the graph rigidity literature is the lack of globally stabilizing control laws that drive the agents to the desired distance. Existing control laws such as the ones proposed in [2],[16] only have local validity for small perturbations around the desired formation, while the control law in [1] refers solely to triangular formations. Motivated by this, in the recent paper [5] we examined the stabilization issue for distance-based formations. A negative gradient control law was proposed based on a potential function between each of the pairs of agents that form an edge in the formation graph. The first result of that paper stated that the system is stabilized to the desired formation provided that the formation graph is a tree. The second result of [5] stated that this was in fact also a necessary condition: the multi-robot system is globally stabilizable to the desired formation with negative gradient control laws if and only if the formation graph is a tree. A summary of the results of [5] is provided here for completeness.

In this paper, we further elaborate on the results of our previous effort and provide additional results on distance based formations. In particular, for the case of cyclic graphs, a characterization of the resulting infinite equilibria of the system is derived relating the edges corresponding to cycles in the formation graph with the ones belonging to its spanning tree. The result further highlights the role of cycles in the equilibria of the system. Furthermore, the control laws are redefined to take into account nonholonomic unicycle type agents.

The rest of the paper is organized as follows: Section II presents the system and formulates the problem treated in this paper, and the necessary mathematical background is presented in Section III. Section IV provides the control law and reviews the results of [5], and proceeds to present the new relation regarding the equilibria of the system in the case of cyclic graphs. Nonholonomic agents are treated in Section V. Simulated examples are included in Section VI while the results are summarized in Section VII.

II. SYSTEM AND PROBLEM STATEMENT

We consider a group of N kinematic agents operating in $\mathbb{R}^2$. Let $q_i \in \mathbb{R}^2$ denote the position of agent i. The configuration space is spanned by $q = [q_1^T, \ldots, q_N^T]^T$. Moreover, each agent $i \in \mathcal{N}$ is assigned a particular orientation $\theta_i \in (-\pi, \pi]$. The objective of the control design is distance-based formation control. Each agent can only communicate with a specific subset $N_i \subset \mathcal{N}$. By convention, $i \notin N_i$. The desired formation can be encoded in terms of an undirected graph, from now on called the *formation graph* $G = \{\mathcal{N}, E\}$, whose set of vertices $\mathcal{N} = \{1, ..., N\}$ is indexed by the team members, and whose set of edges $E = \{(i,j) \in \mathcal{N} \times \mathcal{N} | j \in N_i\}$ contains pairs of vertices that represent inter-agent formation specifications. Each edge $(i,j) \in E$ is

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assigned a scalar parameter $d_{ij} = d_{ji} > 0$, representing the distance at which agents i, j should converge to. Define the set

$$\Phi \triangleq \left\{ q \in \mathbb{R}^{2N} \mid ||q_i - q_j|| = d_{ij},\ \forall (i,j) \in E \right\} \quad (1)$$

of desired distance based formations. The problem is to derive control laws, for which the information available for each agent i is encoded in N_i, that drive the agents to the desired formation, i.e., $\lim_{t \to 0} q(t) = q^* \in \Phi$.

III. Preliminaries

We first review in this section some elements of algebraic graph theory [10] used in the sequel and also present a lemma and a decomposition that will be important for the subsequent analysis.

For an undirected graph G with N vertices the *adjacency matrix* $A = A(G) = (a_{ij})$ is the $N \times N$ matrix given by $a_{ij} = 1$, if $(i,j) \in E$ and $a_{ij} = 0$, otherwise. If there is an edge $(i,j) \in E$, then i, j are called *adjacent*. A *path* of length r from a vertex i to a vertex j is a sequence of $r+1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. For $i = j$, this path is called a *cycle*. If there is a path between any two vertices of the graph G, then G is called *connected*. A connected graph is called a *tree* if it contains no cycles. A *spanning tree* in a connected graph G is a tree subgraph that contains all the vertices of G. An *orientation* on the graph G is the assignment of a direction to each edge. The graph G is called oriented if it is equipped with a particular orientation. The *incidence matrix* $B = B(G) = (B_{ij})$ of an oriented graph is the $\{0, \pm 1\}$-matrix with rows and columns indexed by the vertices and edges of G, respectively, such that $B_{ij} = 1$ if the vertex i is the head of the edge j, $B_{ij} = -1$ if the vertex i is the tail of the edge j, and 0 otherwise. The *Laplacian matrix* is given by $L = BB^T$ [10]. If the graph G contains cycles, then its *cycle space* is the subspace spanned by vectors representing cycles in G [11]. The edges of each cycle in G have a direction, where each edge is directed towards its successor according to the cyclic order. A cycle C is represented by a vector v_C with number of elements equal to the number of edges M of the graph. For each edge, the corresponding element of v_C is equal to 1 if the direction of the edge with respect to C coincides with the orientation assigned to the graph for defining the incidence matrix B, and -1, if the direction with respect to C is opposite to the orientation. The elements corresponding to edges not in C are zero. While L is always positive semidefinite, the matrix $B^T B$ can be either positive semidefinite or positive definite. The next lemma states that in the case of a tree graph, the matrix $B^T B$ is always positive definite:

Lemma 1: If G is tree, then $B^T B$ is positive definite.

Proof: For arbitrary $y \in \mathbb{R}^M$ we have $y^T B^T B y = |By|^2$ and hence $y^T B^T B y > 0$ if and only if $By \neq 0$, i.e., the matrix B has empty null space. For a connected graph, the cycle space of the graph coincides with the null space of B (Lemma 3.2 in [11]). This corresponds to the fact that for G, which has no cycles, zero is not an eigenvalue of B. This implies that $\lambda_{\min}(B^T B) > 0$, i.e., that $B^T B$ is positive definite. $\diamondsuit$

The matrix $B^T B$ was also defined as the "Edge Laplacian" in [21] and its properties were used for providing another perspective to the agreement problem. In this paper, we will use the decomposition of $B^T B$ introduced in [21] to examine the resulting equilibria in the case of formation graphs that contain cycles.

Consider a connected graph G. Similarly to [21], we consider the partition of the incidence matrix

$$B = [\ B_T \quad B_C\ ] \quad (2)$$

where B_T contains the edges of the spanning tree while B_C contains the remaining edges of the graph. From Lemma 1, we know that $B_T^T B_T$ is positive definite.

IV. Control strategy

We provide first in this section the control strategy for single integrator agents introduced in [5] and provide some complementary results. Assume that agents' motion obeys the single integrator model:

$$\dot{q}_i = u_i, i \in \mathcal{N} = \{1, \ldots, N\} \quad (3)$$

where u_i denotes the velocity (control input) for each agent. Denote by $\beta_{ij}(q) = \|q_i - q_j\|^2$ the distance of any pair of agents in the group. The class Γ of formation potentials $\gamma \in \Gamma$ between agents i and j with $j \in N_i$ is defined to have the following properties:

1) $\gamma : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ is a function of the distance between i and j, i.e., $\gamma = \gamma(\beta_{ij})$,
2) $\gamma(\beta_{ij})$ is continuously differentiable,
3) $\gamma(d_{ij}^2) = 0$ and $\gamma(\beta_{ij}) > 0$ for all $\beta_{ij} \neq d_{ij}^2$.

We also define

$$\rho_{ij} \triangleq \frac{\partial \gamma(\beta_{ij})}{\partial \beta_{ij}}$$

Note that $\rho_{ij} = \rho_{ji}$, for all $i, j \in \mathcal{N}, i \neq j$. The proposed control law is

$$u_i = -\sum_{j \in N_i} \frac{\partial \gamma(\beta_{ij}(q))}{\partial q_i} = -\sum_{j \in N_i} 2\rho_{ij}\,(q_i - q_j), i \in \mathcal{N} \quad (4)$$

The set of control laws (4) is written in stack vector form as $u = -2\,(R \otimes I_2)\,q$, where $u = [u_1^T, \ldots, u_N^T]^T$ and the symmetric matrix R is given by

$$R_{ij} = \begin{cases} -\rho_{ij},\ j \in N_i \\ \sum\limits_{j \in N_i} \rho_{ij},\ i = j \\ 0,\ j \notin N_i \end{cases}$$

Consider the candidate Lyapunov function $V(q) = \sum\limits_i \sum\limits_{j \in N_i} \gamma(\beta_{ij}(q))$. Its gradient can be computed as $\nabla V = 4\,(R \otimes I_2)\,q$, so that its time-derivative is given by

$$\dot{V} = -8\,\|(R \otimes I_2)\,q\|^2 \leq 0 \quad (5)$$

The first easy consequence of $\dot{V}$ being negative semidefinite is the following Lemma:

Lemma 2: Consider system (3) driven by the control (4). Then the set of $S_0 = \{q|V(q) < V_0 < \infty\}$ is positively invariant for the trajectories of the closed-loop system.
Proof: This is a consequence of (5). $\diamondsuit$

We next consider the case when the formation potential is given as

$$\gamma\left(\beta_{ij}\left(q\right)\right) = \frac{\left(\beta_{ij} - d_{ij}^2\right)^2}{\beta_{ij}} \tag{6}$$

Note that this potential satisfies all properties of the set Γ. For this case we have

$$\rho_{ij} = \frac{\partial \gamma(\beta_{ij})}{\partial \beta_{ij}} = \frac{\beta_{ij}^2 - d_{ij}^4}{\beta_{ij}^2} \tag{7}$$

The next result involves the fact that with this choice of formation potential, communicating agents do not collide and there is a minimum separation distance between them when the system starts within S_0:

Lemma 3: Consider system (3) driven by the control (4) with γ as in (6), and starting from a set of initial conditions $S_0 = \{q|V(q) < V_0 < \infty\}$. Then it holds that

$$0 < \xi_1 < \beta_{ij}(t) < \xi_2$$

where

$$\xi_{1,2} = \frac{1}{2}\left(2d_{ij}^2 + V_0 \mp \sqrt{4V_0 d_{ij}^2 + V_0^2}\right)$$

for all $(i, j) \in E$ and all $t \geq 0$.
Proof: For every initial condition $q(0) \in S_0$, the time derivative of V remains non-positive for all $t \geq 0$, by virtue of (5). Hence $V(q(t)) \leq V(q(0)) < V_0 < \infty$ for all $t \geq 0$.

Moreover, since $V(q) = \sum\limits_i \sum\limits_{j \in N_i} \gamma(\beta_{ij}(q))$, we have that $\gamma(\beta_{ij}) < V_0$, so that $0 \leq \frac{\left(\beta_{ij} - d_{ij}^2\right)^2}{\beta_{ij}} \leq c$, which implied $\xi_1 < \beta_{ij} < \xi_2$ where $\xi_{1,2} = \frac{1}{2}\left(2d_{ij}^2 + V_0 \mp \sqrt{4V_0 d_{ij}^2 + V_0^2}\right)$. It is easily seen that ξ_1 is strictly positive. $\diamondsuit$

Lemmas 2 and 3, along with LaSalle's Invariance Principle also imply that the system converges to the largest invariant subset of the set $S = \left\{q|\dot{V}(q) = 0\right\}$ which corresponds to $u = -2\left(R \otimes I_2\right) q = 0$, i.e., all agents eventually stop at steady state.

We next review the results of [5] involving stabilization of distance based formations with the control law (4) and γ given as in (6).

Denote by $\bar{q}$ the M-dimensional stack vector of relative position differences of pairs of agents that form an edge in the formation graph, where M is the number of edges, i.e, $M = |E|$ and $\bar{q} = \left[\bar{q}_1^T, \ldots, \bar{q}_M^T\right]^T$, where for an edge $e = (i, j) \in E$ we have $\bar{q}_e = q_i - q_j$.

With simple calculations, we can derive that $\dot{q} = -2\left(R \otimes I_2\right) q$ is equivalent to

$$\dot{\bar{q}} = -\left(B^T BW \otimes I_2\right) \bar{q} \tag{8}$$

where the diagonal matrix W is given by

$$W = 2 \cdot \mathrm{diag}\left\{\rho_e, e \in E\right\} \in \mathbb{R}^{M \times M}$$

Using the previous equation, the convergence properties of the closed-loop system were established in the following theorem in [5]. We review here its proof since it will be useful in the subsequent analysis:

Theorem 4: [5] Assume that the system (3) evolves under the control law (4) with γ as in (6), and that the formation graph is a tree. Then the agents are driven to the desired formation, i.e., $\lim_{t\to\infty} q(t) = q^* \in \Phi$.
Proof (sketch): Since at steady state, $\dot{q} = u = -2\left(R \otimes I_2\right) q = 0$, we also have $\dot{\bar{q}}_e = 0$ for all $e \in E$ and thus $\dot{\bar{q}} = 0$. Then (8) yields $\left(B^T BW \otimes I_2\right) \bar{q} = 0$. By Lemma 1, $B^T B$ is positive definite, and thus $\left(W \otimes I_2\right) \bar{q} = 0$. Since W is diagonal, the last equation is equivalent to $\rho_e \bar{q}_e = 0$ for all $e \in E$. Since ρ_e is scalar this implies $\rho_e = 0$ or $\bar{q}_e = 0$. However, for all $e \in E$ we have $\bar{q}_e(t) \neq 0$ for all $t \geq 0$, by virtue of Lemma 3. We conclude that $\rho_e = 0$ for all $e \in E$ at steady state and hence $\beta_{ij} = d_{ij}^2$, i.e, $||q_i - q_j|| = d_{ij}$ for all $(i, j) \in E$, by virtue of (7). $\diamondsuit$

We next provide the result of [5] that states that the tree structure is a necessary and sufficient condition for global stabilization of distance based formations under the negative gradient control law of the form (4). For any choice of function $\gamma \in \Gamma$, the closed-system dynamics are given by $\dot{q} = u = -2\left(R \otimes I_2\right) q$,, or equivalently by $\dot{\bar{q}} = -\left(B^T BW \otimes I_2\right) \bar{q}$ in the edge space. The analysis leading to Theorem 4 guarantees that $\left(B^T BW \otimes I_2\right) \bar{q} = 0$ at steady state. By virtue of Lemma 1, the matrix $B^T B$ is non-singular only when the formation graph contains no cycles. The following was derived in [5]:

Theorem 5: [5] Assume that the system (3) evolves under the control law (4) and that Φ is non-empty. Consider conditions (i) $u(q) = 0$ only for $q \in \Phi$, and (ii) $\lim_{t\to\infty} q(t) = q^* \in \Phi$. Then there exists a formation potential $\gamma \in \Gamma$ such that (i),(ii) hold if and only if the formation graph is a tree.
Proof (sketch): The "if" part is shown in Theorem 4, with the choice of formation potential field (6). For the "only if part", assume that the closed-loop system has reached a steady state at which $u = 0$. We will show that (i) cannot hold if G is not a tree. If G is not a tree, then $B^T B$ is singular and then the null space of B, and thus $B^T B$, is nonempty. In fact, in this case, using properties of Kronecker products [14], [3], we can show $\left(BW \otimes I_2\right) \bar{q} = 0$. Using the notation $\bar{x}$,$\bar{y}$ for the stack vectors of the elements of $\bar{q}$ in the x and y coordinates, the last equation implies $BW\bar{x} = BW\bar{y} = 0$, i.e., $W\bar{x}, W\bar{y}$ belong to the null space of B. Since G contains cycles, the null space of B is non-empty. Thus we cannot reach the conclusion of the proof of Theorem 4 that $(W \otimes I_2)\bar{q} = 0$. In fact, equations $BW\bar{x} = BW\bar{y} = 0$ have an infinite number of solutions, since $B^T B$ is now singular. Thus condition (i) cannot hold if G is not a tree. We conclude that (i) and (ii) hold only if G is a tree. $\diamondsuit$

A. Cyclic Graphs

In this section we further examine the equilibria of distance-based formations with negative gradient control laws for the case of graphs that contain cycles. Consider the partition (2). Then the edge vector $\bar{q}$ can also be partitioned as

$$\bar{q} = \left[\begin{array}{cc} \bar{q}_T^T & \bar{q}_C^T \end{array}\right]^T \qquad (9)$$

where $\bar{q}_T$ corresponds to the edges of the spanning tree and $\bar{q}_C$ to the remaining ones. Similarly, the matrix $W = 2 \cdot \mathrm{diag}\{\rho_e, e \in E\}$ can be decomposed as

$$W = \left[\begin{array}{cc} W_T & 0 \\ 0 & W_C \end{array}\right]$$

Using (2), we can also compute

$$\begin{aligned} B^T B &= \left[\begin{array}{c} B_T^T \\ B_C^T \end{array}\right] \left[\begin{array}{cc} B_T & B_C \end{array}\right] \\ &= \left[\begin{array}{cc} B_T^T B_T & B_T^T B_C \\ B_C^T B_T & B_C^T B_C \end{array}\right] \end{aligned}$$

so that $\dot{\bar{q}} = -\left(B^T B W \otimes I_2\right)\bar{q}$ can be written as

$$\left[\begin{array}{c} \dot{\bar{q}}_T \\ \dot{\bar{q}}_C \end{array}\right] = (\left[\begin{array}{cc} B_T^T B_T & B_T^T B_C \\ B_C^T B_T & B_C^T B_C \end{array}\right] \left[\begin{array}{cc} W_T & 0 \\ 0 & W_C \end{array}\right] \otimes I_2) \left[\begin{array}{c} \bar{q}_T \\ \bar{q}_C \end{array}\right]$$

or, equivalently

$$\dot{\bar{q}}_T = -(B_T^T B_T W_T \otimes I_2)\bar{q}_T - (B_T^T B_C W_C \otimes I_2)\bar{q}_C \qquad (10)$$

$$\dot{\bar{q}}_C = -(B_C^T B_T W_T \otimes I_2)\bar{q}_T - (B_C^C B_C W_C \otimes I_2)\bar{q}_C \qquad (11)$$

Since $\dot{\bar{q}}_T = \dot{\bar{q}}_C = 0$ at steady state, we get

$$-(B_T^T B_T W_T \otimes I_2)\bar{q}_T - (B_T^T B_C W_C \otimes I_2)\bar{q}_C = 0$$

and since $B_T^T B_T$ is positive definite, we have

$$(W_T \otimes I_2)\bar{q}_T = -((B_T^T B_T)^{-1} B_T^T B_C W_C \otimes I_2)\bar{q}_C \qquad (12)$$

at steady state.

We can further characterize the infinite solutions of equation $\dot{\bar{q}} = -\left(B^T B W \otimes I_2\right)\bar{q}$ in the case of cyclic graphs using (12). For a $l \times l$ matrix M, and $k \leq n$, let $(M)_k$ denote the $k \times n$ matrix that includes the last k rows of M. From the proof of Theorem 4 we know that for each edge e we have either $\rho_e = 0$ at steady state, in the case that this edge has converged to the desired relative distance for the agents that constitute it, or $\rho_e \neq 0$ in the case it has not. Partition now the set of edges corresponding to $\bar{q}_T$ as

$$\bar{q}_T = \left[\begin{array}{cc} \bar{q}_{T_s}^T & \bar{q}_{T_u}^T \end{array}\right]^T$$

where $\bar{q}_{T_s}$ corresponds to the edges that have successfully converged to the desired distance and $\bar{q}_{T_u}$ to the ones that have not. Let $\dim(\bar{q}_{T_s}) = T_s$ and $\dim(\bar{q}_{T_u}) = T_u$. Then the the matrix W_T will have the block diagonal form

$$W_T = \left[\begin{array}{cc} 0 & 0 \\ 0 & W_{T_u} \end{array}\right]$$

since the edges that have converged to the desired values render the corresponding elements of W equal to zero. Morever, W_{T_u} is invertible, since all the elements of this diagonal matrix are nonzero (since they correspond to edges that have not reached the desired distance). Then the following relation holds for $\bar{q}_{T_u}$:

$$(W_{T_u} \otimes I_2)\bar{q}_{T_u} = -(((B_T^T B_T)^{-1} B_T^T B_C W_C)_{T_u} \otimes I_2)\bar{q}_C$$

so that finally

$$\bar{q}_{T_u} = -(W_{T_u}^{-1}((B_T^T B_T)^{-1} B_T^T B_C W_C)_{T_u} \otimes I_2)\bar{q}_C \qquad (13)$$

The last equation provides the relation of all edges that have failed to converge to their desired values at steady state in terms of the cycle edges of the graph.

V. NONHOLONOMIC AGENTS

In this section we modify the control design of the previous sections in order to tackle with nonholonomic kinematic unicycle agents. The control law used in [6] for agreement of multiple nonholonomic agents is redefined in this case to treat distance based formation stabilzation. Agent motion is now described by the following nonholonomic kinematics:

$$\begin{array}{l} \dot{x}_i = u_i \cos\theta_i \\ \dot{y}_i = u_i \sin\theta_i \\ \dot{\theta}_i = \omega_i \end{array}, i \in \mathcal{N} = \{1, \ldots, N\}, \qquad (14)$$

where u_i, ω_i denote the translational and rotational velocity of agent i, respectively.

Define now

$$\gamma_i(q) = \sum_{j \in N_i} \gamma(\beta_{ij}(q))$$

for each agent $i \in \mathcal{N}$. We can now use the control design of [6] for the problem in hand. Specifically, the following discontinuous time-invariant feedback control law is used for each agent i:

$$u_i = -\mathrm{sgn}\{\gamma_{xi}\cos\theta_i + \gamma_{yi}\sin\theta_i\} \cdot \left(\gamma_{xi}^2 + \gamma_{yi}^2\right)^{1/2}, \qquad (15)$$

$$\omega_i = -\left(\theta_i - \theta_{nh_i}\right), \qquad (16)$$

where

$$\gamma_{xi} = \frac{\partial \gamma_i}{\partial x_i} = 2\sum_{j \in N_i} \rho_{ij}\left(x_i - x_j\right)$$

$$\gamma_{yi} = \frac{\partial \gamma_i}{\partial y_i} = 2\sum_{j \in N_i} \rho_{ij}\left(y_i - y_j\right)$$

and $\theta_{nh_i} = \arctan 2\left(\gamma_{yi}, \gamma_{xi}\right)$. Then the following result holds:

Theorem 6: Consider the system of nonholonomic agents (14) with the control law (15),(16). Then the agents are driven to the set

$$S_{nh} = \{(q, \theta) : \left(B^T B W \otimes I_2\right)\bar{q} = 0, \theta_1 = \ldots = \theta_N = 0\}$$

Proof: Using the same steps as in the proof of Theorem 4 in [6], we deduce that the agents converge to a configuration where $\gamma_{xi} = \gamma_{yi} = 0$ for all i with zero orientations. The result now follows from the fact that $\gamma_{xi} = \gamma_{yi} = 0$ for all i implies $2(R \otimes I_2)q = 0$ which furter implies $\left(B^T B W \otimes I_2\right)\bar{q} = 0$. $\diamondsuit$

Hence the control design (15),(16) forces the nonholonomic multi-agent system to behave in exactly the same way as in the single integrator case. Thus, the results regarding the equilibria of the distance based formation controller discussed in the previous sections hold in the nonholonomic case as well.

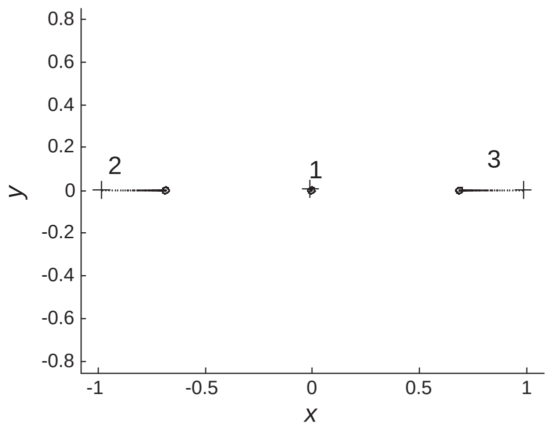

Fig. 1. Example of three single integrator agents from [5]. The resulting configuration belongs to the cycle space of the graph and does not coincide with a point in Φ.

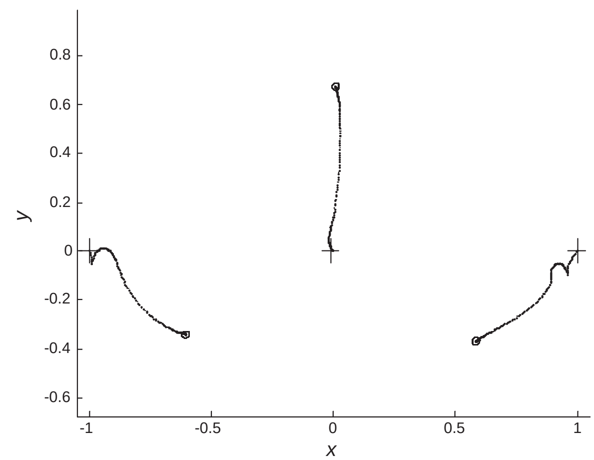

Fig. 2. The three agents now have nonholonomic kinematics and are controlled by (15),(16). The system converges to the desired final formation from the same initial conditions as in the single integrator case.

VI. Simulations

The results of the paper are supported in this section by computer simulations. The purpose of these examples is to show the effect of the nonholonomic kinematics and the communication topology to the resulting equilibria.

In the first simulation we provide a comparison of the single integrator and nonholonomic unicycle cases. We first consider the example taken from [5] where the control law (4) failed to stabilize a system of three single integrator agents to a desired triangular formation. The graph considered is a complete cycle graph,i.e., $N_1 = \{2, 3\}$, $N_2 = \{1, 3\}$, $N_3 = \{2, 3\}$, and $d_{12}^2 = d_{13}^2 = d_{23}^2 = \sqrt{2}$. The agents start from initial positions $q_1(0) = [0, 0]^T$, $q_2(0) = [-1, 0]^T$ and $q_3(0) = [1, 0]^T$. The evolution in the single integrator case is depicted in Figure 1, taken from [5], where the crosses represent the initial positions of the agents and their final locations are noted by a black circle. The system converges to an undesired steady state given by $q_1 = [0, 0]^T$, $q_2 = [-0.6866, 0]^T$ and $q_3 = [0.6866, 0]^T$. The edge distances satisfy $(BW \otimes I_2)\bar{q} = 0$ and $(W \otimes I_2)\bar{q} \neq 0$, and thus the desired formation is not reached. The exact same initial positions are used in Figure 2, where we now consider nonholonomic agents driven by (15),(16). As witnessed in the figure, the agents in the nonholonomic case converge to the desired triangular formation. Thus the undesirable sets of initial conditions that are attractors to the cycle space of the graph G are different than the single integrator case. This is due to the nonholonomic constraints in the agents' motion in the second case.

The next example involves four single integrator agents. In the first example we have a a complete graph and a rectangular formation, to which the agents do indeed converge, as depicted in Figure 3. By deleting the edges between agents 1,3 and 2,4 the resulting equilibria are now shown in Figure 4. In fact, in this example, agents 2 and 4 converge to the same point, since there is no edge and hence no repulsion between them.

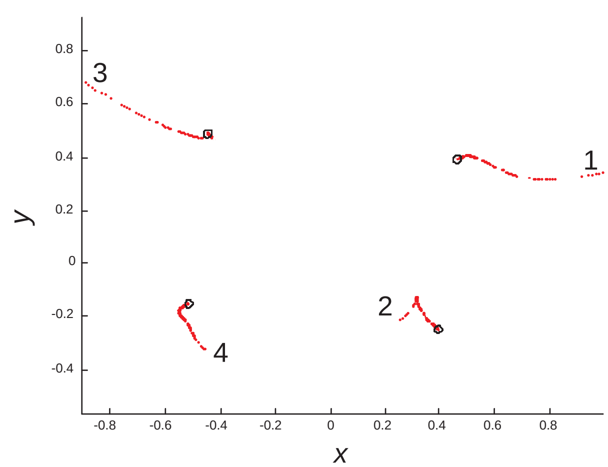

Fig. 3. Four agents with control law (4),(6) and a complete formation graph reach a rectangular formation.

VII. Conclusions

In this paper we provided new results for distance based formation control. In particular, we examined the relation between the cycle space of the formation graph and the resulting equilibria of cyclic formations. Moreover, the results are extended to the case of distance based formation control for nonholonomic agents. Finally, computer simulations supported the derived results.

Future work will focus on further exploring the role of the cycle space and the incidence matrix in other cooperative control problems.

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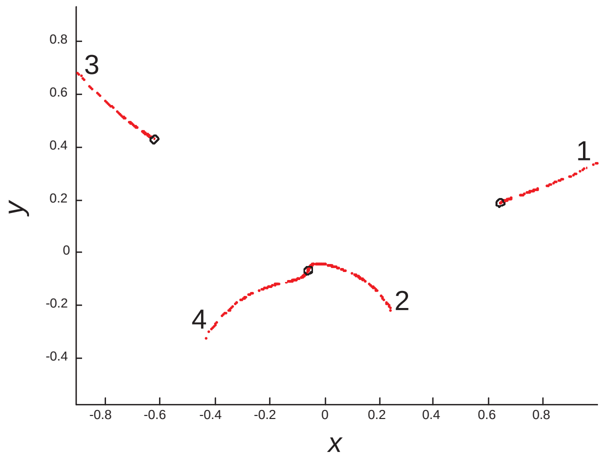

Fig. 4. The edges between agents 1,3 and 2,4 are deleted. The agents end up in a different equilibrium point than the previous case. In fact, agents 2 and 4 converge to the same point, since there is no edge and hence no repulsion between them.

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