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Distance-based Control of Formations with Orientation Control

Myoung-Chul Park† and Hyo-Sung Ahn†

***Abstract*— In this paper, we deal with formation stabilization of a group of mobile agents with orientation control. The proposed method is to combine usual distance-based control with displacement-based control so that the shape of the formation is dominantly controlled by distance-based control, and the orientation of the whole formation is forced to converge to the desired orientation by the additional displacement control input. We assume that the network topology is represented by an undirected graph and that the motion of each agent is simplified to a single-integrator model. We provide stability analysis for the desired formation shape, and four-agent systems are explored to understand asymptotic convergence of the agents. Simulation results are also included to verify our results.**

I. Introduction

Recent publications on distance-based formation control, [1], [2], are mainly focused on the formation shape control problems [3]–[7]. The reason why they deal with only the shape control problems is that the constraints for the desired formation are related only to the inter-agent distances which do not contain orientation information. For example, if we have two agents in the plane, a distance constraint between the agents determines a segment formation that can be freely positioned on the plane with a rotational freedom in addition to two independent translational freedoms. On the other hand, the constraints in displacement-based formation control, [2], [8], are related to relative positions, so the desired formation shape has only the translational freedoms.

Compared to the displacement-based control strategy, although distance-based control cannot maneuver the orientation of the formation, it has an advantage that each agent is not required to have a common sense of the north, i.e., the local reference frames of the agents do not need to be aligned. Thus, in real world implementation, each agent is not required to be equipped with the compass in terms of the formation control algorithm.

Oh and Ahn use the displacement-based formation control algorithm accompanied by an orientation-alignment law [9]. Unlike usual displacement-based control in the literature, the method proposed in [9] does not require that the local reference frames of the agents are aligned from the beginning. Under certain condition on the initial orientation differences, the proposed orientation-alignment law makes the orientation of each agent converge to a common orientation which is not necessarily the orientation of the global reference frame. On the other hand, Cortés uses a sequential initialization algorithm to orient all reference frames [10], and he establishes global formation-shape stabilization with sensing networks having a globally reachable vertex.

Instead of changing the local reference frames, we propose an approach combining the distance- and displacement-based formation control algorithms to achieve the desired shape and the desired orientation of the formation. A fundamental strategy is to use the distance-based formation control algorithm for shape stabilization, and we include additional displacement control terms for only one edge. Then, the displacement control terms perturb the agents so that the relative position corresponding to the edge converges to the desired relative position while the formation shape approaches the desired shape by virtue of the distance control terms.

In the literature, it is known that distance-based formation control with undirected graphs is vulnerable to non-vanishing perturbations [11]–[13]. Belabbas, Mou, Sun and their colleagues show that distance-based control with distance mismatches could result in formations moving permanently. However, we assume that all measurements are correct, and there are no distance mismatches as the authors of the former literature, e.g., [14]–[16], did. The problems which may be caused by non-vanishing perturbations could be dealt with in the future research topics.

Beyond the robustness issues mentioned in [11]–[13], there are many other research topics relevant to formations of mobile agents. For example, flocking [17], formation resizing [18], [19], leader tracking [20], containment control [21], and etc. However, we are interested in shape and orientation control only, so the main content of this paper does not cover those topics.

The rest of the paper is organized as follows. In Section II, we provide basic notation and some definitions used throughout the paper. The concept of graph rigidity and its relevance to distance-based formation control is mentioned, and the control law induced from the proposed potential function is introduced. In Section III, we provide stability analysis for the desired equilibrium set. On the basis of the results in Section III, we explore four-agent systems to understand asymptotic convergence of the agents to the desired formation shape in Section IV. We also provide simulation results to validate our results in Section V. Finally, we summarize our results in Section VI.

II. Preliminaries

In this section, we provide notation and some definitions used in the rest of the paper.

- $\mathbb{R}^n$: n-dimensional Euclidean space
- $\mathbb{R}^{m\times n}$: the set of all m by n real matrices
- $|\mathcal{S}|$: the number of the elements of the finite set $\mathcal{S}$

†School of Mechatronics, Gwangju Institute of Science and Technology (GIST), 123 Cheomdan-gwagiro, Buk-gu, Gwangju, Republic of Korea. E-mail: {mcpark,hyosung}@gist.ac.kr

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- I_n: the n by n identity matrix
- $\mathbf{1}_n \in \mathbb{R}^n$: the vector with all entries equal to 1
- $\mathbf{0}_n \in \mathbb{R}^n$, $\mathbf{0}_{m\times n} \in \mathbb{R}^{m\times n}$: the zero matrices with appropriate size.
- $A \otimes B$: the Kronecker product of matrices A and B.
- $\|\mathbf{x}\|$: the Euclidean norm of a vector $\mathbf{x}$.
- $\mathbf{x}^\perp \in \mathbb{R}^2$: the vector obtained by rotating $\mathbf{x} \in \mathbb{R}^2$ by $\pi/2$.

A. Notation for Graphs, and Graph Rigidity

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph representing the relationships of the agents. The set $\mathcal{V} = \{1,\ldots,N\}$ with $N \geq 2$ denotes the set of all vertices of $\mathcal{G}$, which also means the set of all indices of the agents. We say that an edge (i, j) is in $\mathcal{E}$ if agent i and j are supposed to control the distance between them. In this case, we call i a *neighbor* of j, and vice versa. For convenience, if i and j are neighbors of each other, we use $(\min\{i, j\},\max\{i, j\})$ instead of $(\max\{i, j\},\min\{i, j\})$ to represent the edge connecting i and j, and we define $M = |\mathcal{E}|$ to denote the cardinality of $\mathcal{E}$. From the proposed definition of edge, any self-loop is not allowed, and the graph is *simple*, i.e., $(i,i) \notin \mathcal{E}$ for all $i \in \mathcal{V}$, and (i, j) is the unique edge connecting i and j for all $(i, j) \in \mathcal{E}$. We use $\mathbf{p}_i(t)$ to represent the position vector of vertex (agent) i in $\mathbb{R}^2$ at time t for all $i \in \mathcal{V}$. Then a vector defined by $\mathbf{p} = [\mathbf{p}_1^\top \ \ \ldots \ \ \mathbf{p}_N^\top]^\top \in \mathbb{R}^{2N}$ is called a *realization* of $\mathcal{G}$. We call the pair of $\mathcal{G}$ and its realization $\mathbf{p}$, denoted by $(\mathcal{G},\mathbf{p})$, a *framework*. For two realizations $\mathbf{p}$ and $\mathbf{p}'$ of the same graph $\mathcal{G}$, $(\mathcal{G},\mathbf{p})$ and $(\mathcal{G},\mathbf{p}')$ are said to be *equivalent* if $\|\mathbf{p}_i - \mathbf{p}_j\| = \|\mathbf{p}'_i - \mathbf{p}'_j\|$ for all $(i, j) \in \mathcal{E}$. Two realizations $\mathbf{p}$ and $\mathbf{p}'$ are said to be *congruent* if $\|\mathbf{p}_i - \mathbf{p}_j\| = \|\mathbf{p}'_i - \mathbf{p}'_j\|$ for all $i, j \in \mathcal{V}$.

We define the *rigidity function* of $(\mathcal{G},\mathbf{p})$ as the function $\mathbf{r}\colon \mathbb{R}^{2N} \to \mathbb{R}^M$ given by

$$\mathbf{r}_\mathcal{G}(\mathbf{p}) = [\ldots \quad \|\mathbf{p}_i - \mathbf{p}_j\|^2 \quad \ldots]^\top,\ (i, j) \in \mathcal{E},$$

where the i^{th} component of $\mathbf{r}_\mathcal{G}$ corresponds to the i^{th} edge of $\mathcal{G}$.

Definition 1 (Asimow and Roth (1978), [22]): Let $\mathcal{G}$ be a graph with N vertices and its realization $\mathbf{p}$. Let $\mathcal{K}$ be the complete graph with the same vertex set of $\mathcal{G}$. The framework $(\mathcal{G},\mathbf{p})$ is *rigid* in $\mathbb{R}^2$ if there exists a neighborhood $\mathcal{U}$ of $\mathbf{p}$ in $\mathbb{R}^{2N}$ such that

$$\mathbf{r}_\mathcal{G}^{-1}(\mathbf{r}_\mathcal{G}(\mathbf{p})) \cap \mathcal{U} = \mathbf{r}_\mathcal{K}^{-1}(\mathbf{r}_\mathcal{K}(\mathbf{p})) \cap \mathcal{U}.$$

Intuitively, $(\mathcal{G},\mathbf{p})$ is rigid in $\mathbb{R}^2$ if there exists a neighborhood $\mathcal{U}$ of $\mathbf{p}$ in $\mathbb{R}^{2N}$ such that, for any $\mathbf{p}' \in \mathcal{U}$, equivalence of $(\mathcal{G},\mathbf{p})$ and $(\mathcal{G},\mathbf{p}')$ implies congruence of $\mathbf{p}$ and $\mathbf{p}'$ [2]. In Definition 1, if we can choose $\mathcal{U} = \mathbb{R}^{2N}$, then the framework is *globally rigid* [23].

Consider a matrix $R_\mathcal{G} \in \mathbb{R}^{M\times(2N)}$ defined by

$$R_\mathcal{G}(\mathbf{p}) = \frac{1}{2}\frac{\partial \mathbf{r}_\mathcal{G}}{\partial \mathbf{p}}.$$

We call $R_\mathcal{G}$ the *rigidity matrix* of $(\mathcal{G},\mathbf{p})$. If there is no confusion, we remove the subscript $\mathcal{G}$ for convenience.

Definition 2 (Asimow and Roth (1979), [24]): With the same notation used in Definition 1, $(\mathcal{G},\mathbf{p})$ is *infinitesimally rigid* in $\mathbb{R}^2$ if the null space of $R(\mathbf{p})$ is equal to the tangent space to $\mathbf{r}_\mathcal{K}^{-1}(\mathbf{r}_\mathcal{K}(\mathbf{p}))$ at $\mathbf{p}$.

There are subtle differences between rigidity and infinitesimal rigidity. Depending on the realization $\mathbf{p}$ of $\mathcal{G}$, $(\mathcal{G},\mathbf{p})$ may not be infinitesimally rigid even if it is rigid. However, if $(\mathcal{G},\mathbf{p})$ is infinitesimally rigid, then it is rigid [24].

There is a useful theorem providing a necessary and sufficient condition to determine whether or not a given framework is infinitesimally rigid.

Theorem 1 ([24], [25]): A framework $(\mathcal{G},\mathbf{p})$ with $N \geq 2$ vertices is *infinitesimally rigid* in $\mathbb{R}^2$ if and only if the rank of the rigidity matrix of $(\mathcal{G},\mathbf{p})$ is equal to $2N-3$.

The relevance between graph rigidity and the distance constraints for formations is explained in [1]. Although using rigidity is enough to characterize the unique formation shape in a local sense, we need infinitesimal rigidity for further analysis. Thus we assume that the framework representing the desired formation shape is infinitesimally rigid. For the results established on the basis of rigidity assumption, refer to [26], which deals with the distance-based formation control problems of rigid frameworks.

For later use, we invoke a proposition on the loss of infinitesimal rigidity.

Lemma 1: Consider a graph $\mathcal{G}$ with $N \geq 3$ and its realization $\mathbf{p}$. Suppose that $\mathcal{G}$ has a vertex which has exactly two neighbors, and let i denote the vertex. Let j and $k \neq j$ denote the neighbors of i. If $(\mathbf{p}_i - \mathbf{p}_j)$ and $(\mathbf{p}_i - \mathbf{p}_k)$ are linearly dependent, then $(\mathcal{G},\mathbf{p})$ is not infinitesimally rigid in $\mathbb{R}^2$.

Proof: Suppose that $(\mathcal{G},\mathbf{p})$ is infinitesimally rigid. Then, from Theorem 1, the nullity of $R(\mathbf{p})$ is equal to 3. If all of the vertices are not collocated at the same point, then a basis of the null space of $R(\mathbf{p})$ is given by

$$\mathcal{B} = \left\{ \mathbf{1}_N \otimes \begin{bmatrix}1\\0\end{bmatrix},\ \mathbf{1}_N \otimes \begin{bmatrix}0\\1\end{bmatrix},\ \begin{bmatrix}\mathbf{p}_1^\perp\\ \vdots \\ \mathbf{p}_N^\perp\end{bmatrix} \right\}.$$

If all of the vertices are collocated, then obviously the framework is not infinitesimally rigid, so we discard the possibility of collocation. The basis vectors in $\mathcal{B}$ correspond to the trivial infinitesimal motions (two independent translations and one rotation) [25]. If we can find a nontrivial infinitesimal motion, we can reach a contradiction. Such a nontrivial infinitesimal motion $\mathbf{v}$ is given by

$$\mathbf{v} = \begin{cases} \begin{bmatrix}\mathbf{0}_{2(i-1)}^\top & \mathbf{1}_2^\top & \mathbf{0}_{2(N-i)}^\top\end{bmatrix}^\top, & \text{if } \mathbf{p}_i = \mathbf{p}_j = \mathbf{p}_k, \\ \begin{bmatrix}\mathbf{0}_{2(i-1)}^\top & [(\mathbf{p}_i - \mathbf{p}_l)^\perp]^\top & \mathbf{0}_{2(N-i)}^\top\end{bmatrix}^\top, & \text{otherwise,} \end{cases}$$

where $l \in \{m \in \{j,k\} \mid \mathbf{p}_i - \mathbf{p}_m \neq \mathbf{0}_2\}$. Note that, for infinitesimal rigidity, at least $(2N-3)$ row vectors of $R(\mathbf{p})$ must not be the zero vector. In that case, $\mathbf{v}$ is linearly independent from the vectors in $\mathcal{B}$. Thus, the nullity of $R(\mathbf{p})$ exceeds 3, which means that $(\mathcal{G},\mathbf{p})$ is not infinitesimally rigid. Note that $R(\mathbf{p})\mathbf{v} = \mathbf{0}_M$ from the fact that $(\mathbf{p}_i - \mathbf{p}_j)$ and $(\mathbf{p}_i - \mathbf{p}_k)$ are linearly dependent, and i has exactly two neighbors. ■

B. Equations of Motion

We assume that each agent follows the integrator model given by

$$\dot{\mathbf{p}}_i = \mathbf{u}_i,\ \forall i \in \mathcal{V}.$$

Thus, we can assign any piecewise continuous and bounded velocity vector to each agent. Let $\bar{\mathbf{p}} \in \mathbb{R}^{2N}$ be a representative realization of the desired formation shape. We use e_{ij} to denote the squared-distance errors defined by

$$e_{ij} = \|\mathbf{p}_i - \mathbf{p}_j\|^2 - \|\bar{\mathbf{p}}_i - \bar{\mathbf{p}}_j\|^2,\ \forall (i,j) \in \mathcal{E},$$
$$\mathbf{e} = [\dots\ \ e_{ij}\ \ \dots]^\top \in \mathbb{R}^M.$$

For convenience, we assume that $(1,2) \in \mathcal{E}$ without loss of generality. If $(1,2) \notin \mathcal{E}$, we can reorder the indices of the agents so that $(1,2) \in \mathcal{E}$. Then, consider a potential function $V\colon \mathbb{R}^{2N} \to \mathbb{R}$ given by

$$V(\mathbf{p}) = \phi(\mathbf{p}) + \varphi(\mathbf{p}),$$

where

$$\phi(\mathbf{p}) = \frac{1}{4}\mathbf{e}^\top\mathbf{e}, \quad \varphi(\mathbf{p}) = \frac{1}{2}\tilde{\mathbf{p}}_{12}^\top\tilde{\mathbf{p}}_{12}, \tag{1}$$

and $\tilde{\mathbf{p}}_{12} = (\mathbf{p}_1 - \mathbf{p}_2) - (\bar{\mathbf{p}}_1 - \bar{\mathbf{p}}_2)$. Hence, V is obtained by modifying the potential functions used in [14], [15], [27] with the additional term of the displacement error.

In usual distance-based formation control found in [14], [15], φ is equal to zero, and the agents are controlled so that $V \to 0$ as $\mathbf{p}(t)$ approaches a point that is congruent to $\bar{\mathbf{p}}$. However, that does not answer whether or not the orientation of the formation also approaches the orientation of the desired formation shape. In that regard, we use nonzero φ defined in (1) for orientation control.

The control law derived from the gradient-descent algorithm is given by

$$\dot{\mathbf{p}} = \mathbf{u} = -\nabla V(\mathbf{p}) = -[R(\mathbf{p})]^\top\mathbf{e} + \begin{bmatrix} -\tilde{\mathbf{p}}_{12} \\ \tilde{\mathbf{p}}_{12} \\ \mathbf{0}_{2(N-2)} \end{bmatrix}. \tag{2}$$

Let $\mathcal{D}_p = \{\mathbf{x} \in \mathbb{R}^{2N} \mid V(\mathbf{x}) = 0\}$. Since $V(\mathbf{p})$ is non-negative and continuously differentiable, we have $\nabla V(\mathbf{p}) = \mathbf{0}_{2N}$ for any $\mathbf{p} \in \mathcal{D}_p$. Thus $\mathcal{D}_p$ is an equilibrium set for (2).

Unfortunately, $\mathcal{D}_p$ is not bounded, which causes some difficulties in analyzing stability of $\mathcal{D}_p$. To avoid such difficulties, Dörfler and Francis use the *link dynamics* [16], and Krick et al. use the centroid decomposition [14]. Instead, we are going to use a decomposition procedure used in [27] which is similar to one used in [14].

III. Stability of the Desired Equilibrium Set

A. Decomposition of the Centroid Dynamics

From the fact that $R(\mathbf{p})(\mathbf{1}_N \otimes I_2) = \mathbf{0}_{M\times 2}$, we can see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{p}_i\right) = -\frac{1}{N}(\mathbf{1}_N^\top \otimes I_2)\nabla V(\mathbf{p}) = \mathbf{0}_2,$$

Therefore, once an initial condition is given, the centroid of the formation does not move along the solution trajectory for (2). Let $\mathbf{m} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{p}_i$, and $\mathbf{q}_i = \mathbf{p}_i - \mathbf{m}$, $\forall i \in \mathcal{V}$. Let $\mathbf{q} = [\mathbf{q}_1^\top\ \ \dots\ \ \mathbf{q}_N^\top]^\top$. Consider an invertible linear transformation $\mathscr{L}\colon \mathbb{R}^{2N} \to E_0 \times \mathbb{R}^2$, $\mathbf{p} \mapsto (\mathbf{q}_1, \dots, \mathbf{q}_N, \mathbf{m})$, where $E_0 = \{\mathbf{x} \in \mathbb{R}^{2N} \mid (\mathbf{1}_N^\top \otimes I_2)\mathbf{x} = \mathbf{0}_2\}$. Since e_{ij} is a function of relative position vectors for each $(i,j) \in \mathcal{E}$, we can define the potential function in terms of $\mathbf{q}$ as $\bar{V}(\mathbf{q}) = V(\mathbf{p})$. From the fact that $\partial V(\mathbf{p})/\partial\mathbf{p} = \partial\bar{V}(\mathbf{q})/\partial\mathbf{q}$, and the centroid is stationary, we can decompose the original systems (2) into the following systems;

$$\begin{aligned} \dot{\mathbf{q}} &= -\left[I_{2N} - \frac{1}{N}(\mathbf{1}_N\mathbf{1}_N^\top) \otimes I_2\right]\nabla\bar{V}(\mathbf{q}) \\ &= -\nabla\bar{V}(\mathbf{q}), \\ \dot{\mathbf{m}} &= \mathbf{0}_2. \end{aligned} \tag{3}$$

Consequently, we can investigate stability of the desired formation by analyzing (3) instead of (2).

As we define $\bar{V}$ by $\mathbf{q}$, the rigidity matrix can be calculated by $\mathbf{q}$ instead of $\mathbf{p}$ since the rigidity matrix is determined by the relative position vectors of the agents. Thus, we can define the rigidity matrix in terms of $\mathbf{q}$ as $\bar{R}(\mathbf{q})$ such that $\bar{R}(\mathbf{q}) = R(\mathbf{p})$.

Let $\mathcal{D}_q = \{\mathbf{x} \in E_0 \mid \bar{V}(\mathbf{x}) = 0\}$. It is clear that $\mathbf{p}$ is in $\mathcal{D}_p$ if and only if $\mathbf{q}$ is in $\mathcal{D}_q$. Unlike $V(\mathbf{p})$, $\bar{V}(\mathbf{q})$ is radially unbounded, so $\mathcal{D}_q$ is compact, and we can consider another compact set Ω_q defined by $\Omega_q = \{\mathbf{x} \in E_0 \mid \bar{V}(\mathbf{x}) \le c,\ c > 0\}$. From (3), we have $\dot{\bar{V}}(\mathbf{q}) = -\|\nabla\bar{V}(\mathbf{q})\|^2 \le 0$ for any $\mathbf{q}$ in E_0, which means that Ω_q is positively invariant, and $\mathcal{D}_q$ is stable.

B. Convergence of the Solution Trajectories

Although we know that $\mathcal{D}_q$ is stable, it is not yet revealed whether or not $\mathbf{q}(t)$ converges to a point, which means that the existence of $\lim_{t\to\infty}\mathbf{p}(t)$ has yet to be guaranteed. To answer the question, we invoke the following lemma;

Lemma 2 (Theorem 1 in [27]): Let $\mathcal{M}$ denote a real analytic manifold endowed with a real analytic Riemannian metric. Let f denote an arbitrary real analytic function on a real analytic Riemannian manifold $\mathcal{M}$ such that all sublevel sets $\{\mathbf{y} \in \mathcal{M} \mid f(\mathbf{y}) \le c,\ c > 0\}$ are compact. Then each solution of $\dot{\mathbf{x}} = -\nabla f(\mathbf{x})$ exists for all $t \ge 0$ and converges for $t \to +\infty$ to a single equilibrium point $\mathbf{x}^*$ which satisfies $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

From Lemma 2, we can state the following conclusion;

Corollary 1: Let $\mathcal{Q}_q = \{\mathbf{x} \in E_0 \mid \nabla\bar{V}(\mathbf{x}) = \mathbf{0}_{2N}\}$. Then $\mathbf{q}(t)$, which is the solution trajectory for (3), converges to a point in $\mathcal{Q}_q$ as $t \to +\infty$. Equivalently, $\mathbf{p}(t)$, which is the solution trajectory for (2), also converges to a point in $\{\mathbf{x} \in \mathbb{R}^{2N} \mid \nabla V(\mathbf{x}) = \mathbf{0}_{2N}\}$ as $t \to +\infty$.

IV. Four-agent Systems

In this section, we explore a particular example on a group of four mobile agents. General results beyond this particular example can be found in [28].

Consider a formation represented by $(\mathcal{G}, \mathbf{p})$ of which the vertex set $\mathcal{V}$ is $\{1,2,3,4\}$, and the edge set $\mathcal{E}$ is $\{(1,2),(1,3),(2,3),(2,4),(3,4)\}$ (see Fig. 1). Then (2) is

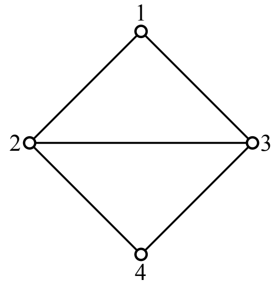


Fig. 1. Four-vertex graph which is minimally rigid in $\mathbb{R}^2$.

written in detail by

$$\dot{\mathbf{p}} = -\nabla V(\mathbf{p}) = -\nabla \bar{V}(\mathbf{q}) = -\underbrace{\begin{bmatrix} \mathbf{p}_{12} & \mathbf{p}_{13} & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{p}_{21} & \mathbf{0}_2 & \mathbf{p}_{23} & \mathbf{p}_{24} & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{0}_2 & \mathbf{p}_{34} \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{p}_{42} & \mathbf{p}_{43} \end{bmatrix}}_{=R(\mathbf{p})=\bar{R}(\mathbf{q})} \begin{bmatrix} e_{12} \\ e_{13} \\ e_{23} \\ e_{24} \\ e_{34} \end{bmatrix} + \begin{bmatrix} -\tilde{\mathbf{p}}_{12} \\ \tilde{\mathbf{p}}_{12} \\ \mathbf{0}_2 \\ \mathbf{0}_2 \end{bmatrix}, \quad (4)$$

where $\mathbf{p}_{ij} = \mathbf{p}_i - \mathbf{p}_j$ for all $i, j \in \mathcal{V}$.

From Corollary 1, we know that $\mathbf{p}(t)$ converges to the equilibrium set of (4). However, it does not mean that the potential function also converges to 0. Therefore, we need further analysis to investigate asymptotic stability.

Theorem 2: For the four-agent formation described in Fig. 1, assume that the desired formation shape is given by $\bar{\mathbf{p}}$ and that $(\mathcal{G}, \bar{\mathbf{p}})$ is infinitesimally rigid. Then $\mathcal{D}_q$ is (locally) asymptotically stable, and $\mathbf{q}(t)$ converges to a point in $\mathcal{D}_q$. Equivalently, $\mathbf{p}(t)$ also converges to a point in $\mathcal{D}_p$.

Proof: We already know that $\mathcal{D}_q$ is stable. Consider Ω_q and the constant c used in the definition of Ω_q. We can take c small enough so that $\bar{R}(\mathbf{q})$ has full row rank for all $\mathbf{q} \in \Omega_q$ by virtue of the properties of infinitesimal rigidity. To reveal asymptotic stability of $\mathcal{D}_q$, let us first characterize $\mathcal{Q}_q \cap \Omega_q$. For any $\mathbf{q} \in \mathcal{Q}_q \cap \Omega_q$, $\mathbf{p}_{42}$ and $\mathbf{p}_{43}$ are linearly independent from the contrapositive of Lemma 1, which results in that $e_{24} = e_{34} = 0$ from $\nabla \bar{V} = \mathbf{0}_{2N}$. As well, $\mathbf{p}_{12}$ and $\mathbf{p}_{13}$ are linearly independent, which is equivalent that $\mathbf{p}_{31}$ and $\mathbf{p}_{32}$ are linearly independent. Hence, e_{13} and e_{23} are also equal to 0. Now we have

$$-\mathbf{p}_{12} e_{12} = -\mathbf{p}_{12} + \bar{\mathbf{p}}_{12}, \quad (5)$$

from (4). Note that $\mathbf{p}_{12}$ and $\bar{\mathbf{p}}_{12}$ must be linearly dependent. Let $p_{12} = \|\mathbf{p}_{12}\| \geq 0$, and $\bar{p}_{12} = \mathbf{p}_{12}^\top \bar{\mathbf{p}}_{12} / p_{12}$. Then, from (5), we have

$$(p_{12}^2 + \bar{p}_{12} p_{12} - 1)(p_{12} - \bar{p}_{12}) = 0, \quad (p_{12} \neq 0), \quad (6)$$

and $p_{12} = \bar{p}_{12}$ is an isolated solution of (6). Thus, we can choose $c = c_1$ with sufficiently small c_1 so that $\bar{p}_{12} > 0$. Moreover, by taking $c = c_2 \leq c_1$ with sufficiently small c_2, we can choose Ω_q so that $p_{12} = \bar{p}_{12}$ is the only solution of (6) for $\mathbf{q} \in \Omega_q$. Thus, for any $\mathbf{q} \in \mathcal{Q}_q \cap \Omega_q$, $\nabla \bar{V} = \mathbf{0}_{2N}$ is equivalent to $[\mathbf{e}^\top \ \ \tilde{\mathbf{p}}_{12}^\top] = \mathbf{0}_{M+2}^\top$. Therefore, it is true that $\mathcal{Q}_q \cap \Omega_q = \mathcal{D}_q$. Since Ω_q is positively invariant, $\mathbf{q}(t)$ converges to a point in $\mathcal{D}_q$ from Corollary 1, which means that $\mathbf{p}(t)$ converges to a point in $\mathcal{D}_p$. ■

Consequently, from Theorem 2, we can conclude that if the initial formation shape and the desired formation shape are close enough, and the initial orientation and the desired orientation are close enough also, then the agents finally achieve the desired formation shape with desired orientation prescribed by $\bar{\mathbf{p}}$.

Remark 1: Under distance-based formation control, it is known that the final formation shape may not be congruent to the desired shape if the initial formation shape and the desired formation shape are not close enough. Such characteristics are called *flip ambiguity* and *flex ambiguity* [1].

Remark 2: There exists an incorrect equilibrium set in which the potential function does not converge to 0 [14], [15]. For example, if $(\mathbf{p}_1(0) - \mathbf{p}_2(0))$ is parallel to $(\bar{\mathbf{p}}_1 - \bar{\mathbf{p}}_2)$ with $\mathbf{p}_3(0)$ and $\mathbf{p}_4(0)$ located on the line connecting $\mathbf{p}_1(0)$ and $\mathbf{p}_2(0)$, then $\mathbf{q}(t)$ approaches $\mathcal{Q}_q \setminus \mathcal{D}_q$. Hence, our results on stability and asymptotic stability are valid only in a local sense.

Remark 3: As we mentioned in Section I, an agent is not necessarily required to be equipped with the compass if the agent is governed by the distance-based control algorithm. Thus, in our problem, only agents 1 and 2 need to know the orientation of the global reference frame, whereas the other agents do not.

V. Simulation

We provide simulation results of the four-agent systems explored in Section IV. From Theorem 2, we know that if the initial formation shape and the desired formation shape are close enough in the sense that $V(\mathbf{p}(0))$ is small enough, then $\mathbf{p}(t)$ converges to $\mathcal{D}_p$ as $t \to +\infty$. To compare the effect of orientation control with the results in the literature, we show the trajectories of the agents in Fig. 2(a) provided that the agents are governed by the control law derived with $\varphi = 0$. The desired formation shape is given in Fig. 1, and we have

$$\bar{\mathbf{p}} = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}^\top.$$

In Fig. 2(a), we can observe that the orientation of the final formation is different from the orientation of the desired formation although the shapes are congruent.

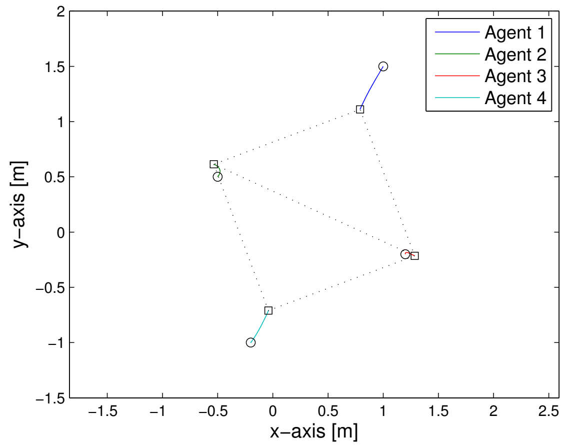
In Fig. 2(b), the trajectories of the agents governed by (4) (with nonzero φ defined in (1)) are drawn. The initial condition is the same as one used in Fig. 2(a), which is given by

$$\mathbf{p}(0) = \begin{bmatrix} 1 & 1.5 & -0.5 & 0.5 & 1.2 & -0.2 & -0.2 & -1 \end{bmatrix}^\top.$$

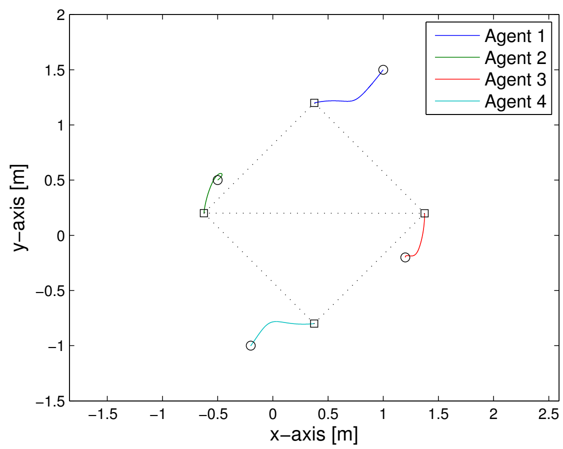
Corresponding history of $V(\mathbf{p}(t))$ is shown in Fig. 2(c), and we can observe that the potential function finally converges to 0.

Although we do not provide analysis on six-agent systems in detail, we show the simulation results of six-agent complete formation[1] in Fig. 4. The desired formation shape is

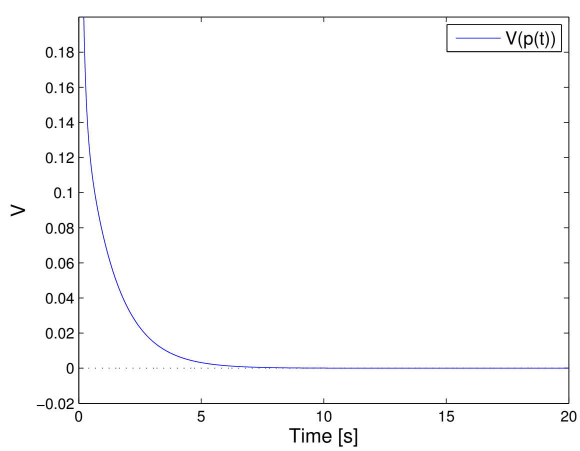
[1]By complete formation, we mean a formation of which the underlying graph is complete.



(a) Without orientation control



(b) With orientation control



(c) $V(\mathbf{p}(t))$ on $[0, 20]$

Fig. 2. Simulation of four-agent minimally rigid formation

shown in Fig. 3 where

$$\bar{\mathbf{p}} = \begin{bmatrix} 2 & 0 & 1 & \sqrt{3} & -1 & \sqrt{3} & -2 & 0 & -1 & -\sqrt{3} & 1 & -\sqrt{3} \end{bmatrix}^\top.$$

Without orientation control, only the formation shape approaches the desired shape (Fig. 4(a)). However, as we revealed in four-agent case, with orientation control, the agents achieve the desired orientation as well as the desired shape (Fig. 4(b)).

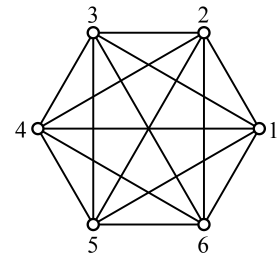


Fig. 3. Six-vertex complete graph which is globally rigid.

VI. Conclusion

We proposed a control strategy that can maneuver the mobile agents in the plane so that they achieve not only the desired formation shape but also the desired orientation of the formation. The fundamental idea for formation shape control is using the distance control strategy proposed in the previous literature [1], [2], [14]–[16], [27]. By adding a displacement error term to the potential function, the whole formation is forced by the proposed control law to achieve the same orientation of the desired realization as well as the desired formation shape. On the basis of our analysis, we provided some simulation results to assert our conclusion.

From the proposed control law, we can observe the possibility that combination of distance and displacement constraints results in new characteristics compared to the existing results in the literature. We expect that we would be able to mix those different constraints, thereby the rigidity condition may be relaxed.

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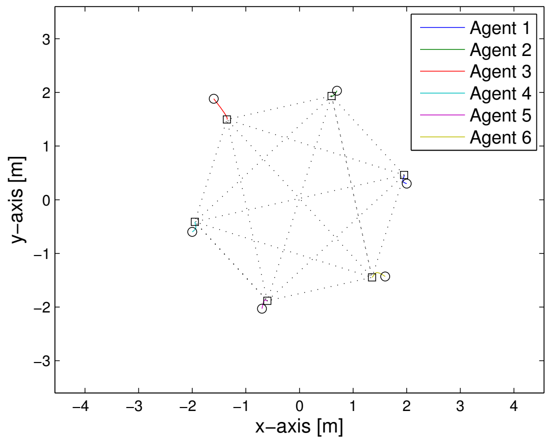
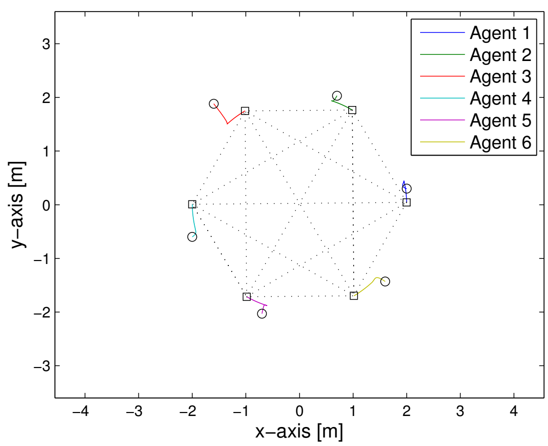
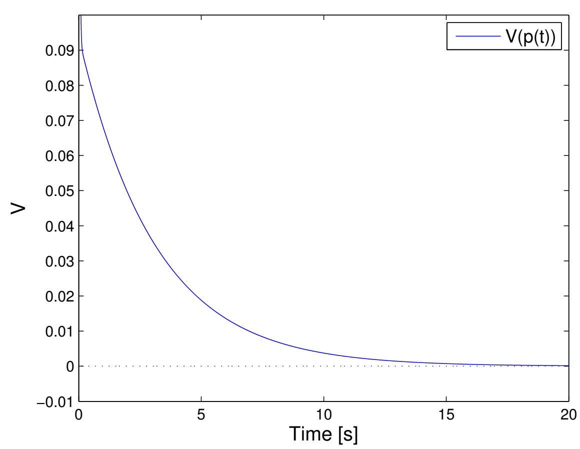


(a) Without orientation control



(b) With orientation control



(c) $V(\mathbf{p}(t))$ on $[0,20]$

Fig. 4. Simulation of six-agent complete formation

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