2014 American Control Conference (ACC)
June 4-6, 2014. Portland, Oregon, USA

Multi-Agent Formation Maneuvering and Target Interception with Double-Integrator Model

Xiaoyu Cai[1] and Marcio de Queiroz[2]

Abstract— **This paper introduces distance-based control laws for the multi-agent formation maneuvering and target interception problems using a double-integrator agent model and rigid graph theory. The proposed controls consist of a formation acquisition term, dependent on the graph rigidity matrix, and a formation maneuvering or target interception term. The control laws are only a function of the relative position/velocity of agents in an infinitesimally and minimally rigid graph, the agent's own velocity, and either the desired velocity of the formation or the target's relative position to the leader and velocity. The target interception control includes a variable structure-type term to compensate for the unknown target acceleration. A Lyapunov-based stability analysis is used to prove that the control objectives are met.**

I. INTRODUCTION

A multi-agent system refers to a network of interacting "agents" that collectively perform a complex task. The concept is inspired by the collective behavior of biological systems in nature, e.g., flock of birds, school of fish, and colonies of bees. Interestingly, the behavior of such biological swarms is distributed and decentralized as each biological agent operates using its own local sensing and control mechanisms devoid of global knowledge or planning [11]. An example of an engineering multi-agent system is a group of autonomous (ground, underwater, water surface, or air) vehicles performing surveillance, reconnaissance, mapping, or search of an area. Recent advances in sensor technology, computer processing, communication systems, and power storage now make it feasible to deploy such swarms of coordinated, cooperating vehicles in various environments. Multi-agent systems offer many advantages over a large single agent such as more efficient and complex task execution, robustness when one or more agents fail, scalability, versatility, adaptability, and lower cost. However, they introduce a host of unique challenges: coordination and cooperation among agents, distribution of information and subtasks, communication protocols, design of control laws, and collision avoidance.

Among the many coordination and control problems for multi-agent systems, we are interested here in the class of *formation* problems. Specifically, our focus is on three related problems with increasing level of complexity: *formation acquisition*, which is defined as designing control inputs for the agents so that they form a pre-defined geometric shape in space; *formation maneuvering*, where agents are required to simultaneously acquire a formation and move cohesively following a pre-defined (time-varying) trajectory; and *target interception*, where agents intercept and surround a moving target with a pre-defined formation. Note that formation acquisition is a pre-condition for formation maneuvering and target interception.

The aforementioned control problems are relatively straightforward to solve when the agents' global coordinates (absolute positions) are available via a central planner. However, as pointed out in [15], a global positioning system (GPS), which is typically used in such cases, has limited accuracy when there is no line of sight between the GPS receiver and satellite (e.g., urban areas, dense vegetation, and dense clouds). Therefore, we consider here the decentralized formation problem where each agent has only locally-sensed information about the other agents obtained from onboard sensors, such as an inertial-type navigation system, laser range finder, camera, and/or compass.

Graph theory, specifically the concepts of graph Laplacian and graph rigidity, is a natural tool for describing the multi-agent formation shape and the inter-agent sensing and communication network topology in the decentralized case. Rigid graph theory, in particular, naturally ensures that the inter-agent distance constraints of the desired formation are enforced through the graph rigidity. This implicitly ensures that collisions between agents are avoided while acquiring the formation. Another advantage of using the inter-agent distances as the controlled variables is that position measurements in a global coordinate frame are not required [26] as would be the case when employing "consensus" algorithms [23]. An overview of rigid graph theory and its application to sensing, communication, and control architectures for formations of autonomous vehicles was presented in Ref. [1].

In this paper, we propose control laws for the formation maneuvering and target interception problems of n planar agents using a double-integrator model for their motion. In solving these problems, we model the formation with an undirected rigid graph and consider the stabilization of the inter-agent distance dynamics to desired distances. We build upon our previous result in [5], which was based on the simpler, single-integrator (kinematic) model. Specifically, we exploit the backstepping technique [16] to construct rigidity matrix-based control laws augmented with a term to enable the agents to perform formation maneuvering or target interception, simultaneously with formation acquisition, for the double-integrator model. We begin with the formation

[1]X. Cai is with Baker Hughes, 11211 FM 2920, Tomball, TX, 77375 USA; xiaoyucai2008@gmail.com.

[2]M. de Queiroz is with the Department of Mechanical and Industrial Engineering, Louisiana State University, Baton Rouge, LA 70803 USA; dequeiroz@me.lsu.edu (corresponding author).

maneuvering problem, where the desired swarm (group) motion is known to all agents. We then show how the idea behind the control design for formation maneuvering can be extended to the target interception problem. In this case, we use the leader-follower concept by assigning the leader role to one agent in the formation, who is responsible for intercepting the moving target. We assume the target's relative position to the leader and velocity are known and can be broadcast to the followers; however, the target's acceleration is *unknown* to all agents. To deal with this uncertainty, the target interception component of the control law will contain a variable structure-type term to compensate for the unknown target acceleration. The graph that models the desired formation in this case is constructed such that the leader is in the convex hull of the followers. As a result, the proposed control ensures the followers eventually enclose the target. Our stability analysis for both problems relies on rigid graph theory and Lyapunov-based arguments, and provides exponential formation acquisition. As a result, asymptotic formation maneuvering or target interception can be readily proven.

Previous results on formation acquisition based on controlling inter-agent distances for the single- and double-integrator models can be found in [4], [7], [9], [15], [18], [26] and [6], [20], respectively. In [19], a distance-based formation maneuvering controller was proposed using the single-integrator model for cycle-free persistent formations under the condition that the trajectory velocity is sufficiently low. A relative position-based formation maneuvering protocol was introduced in [27] for the single-integrator model that ensures formation acquisition in finite time. A collaborative target tracking controller based on flocking and Kalman-type filtering algorithms was given in [22] using the double-integrator agent model. In [11], distance-based formation maneuvering and target interception schemes were designed using the single-integrator model by adding terms to a gradient-of-potential-function law. In one of the target interception schemes in [11], the absolute velocity of the target is uncertain but with known bound. The double-integrator model was used in [21] where maneuvering of the flocking agents was achieved by adding a dynamic virtual leader-dependent term to the control scheme. Recently, [17] used an iterative learning controller to ensure finite-time formation maneuvering with bounded tracking error.

The main contribution of this paper is that it is the first to demonstrate how to extend the rigid graph-based control framework [4], [9], [15], [18], [26] to the formation maneuvering and target interception problems using the double-integrator dynamic model for the agents.

II. PRELIMINARIES

Some basic concepts of rigid graph theory in $\mathbb{R}^2$ used by our control formulation are outlined below.

An undirected graph G is a pair (V, E) where $V = \{1, 2, \ldots, n\}$ is the set of vertices and $E \subset V \times V$ is the set of undirected edges such that if vertex pair $(i, j) \in E$ then so is (j, i). The number of edges l is given by $l \in \{1, \ldots, n(n-1)/2\}$. Let the set of neighbors of vertex i be denoted by

$$\mathcal{N}_i(E) = \{j \in V \mid (i, j) \in E\}. \tag{1}$$

If $p_i \in \mathbb{R}^2$ is the coordinate of vertex i, then a framework F is a pair (G, p) where $p = (p_1, \ldots, p_n) \in \mathbb{R}^{2n}$. That is, a framework is simply a realization of the graph at given points in the plane. Based on an arbitrary ordering of the edges in E, the edge function $\phi : \mathbb{R}^{2n} \to \mathbb{R}^l$ is given by

$$\phi(p) = \left(\ldots, \|p_i - p_j\|^2, \ldots\right), \quad (i, j) \in E \tag{2}$$

where $\|\cdot\|$ denotes the Euclidean norm. The kth component of (2), $\|p_i - p_j\|^2$, corresponds to the kth edge in E connecting vertices i and j. The rigidity matrix $R : \mathbb{R}^{2n} \to \mathbb{R}^{l \times 2n}$ of $F = (G, p)$ is defined as

$$R(p) = \frac{1}{2}\frac{\partial \phi(p)}{\partial p}. \tag{3}$$

It is known that $\text{rank}[R(p)] \leq 2n - 3$ [2].

An isometry of $\mathbb{R}^2$ is a bijective map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that [12]

$$\|x - y\| = \|T(x) - T(y)\|, \quad \forall x, y \in \mathbb{R}^2. \tag{4}$$

Note that T accounts for rotation, translation, and reflection of the vector $x - y$. We denote the set of all isometric frameworks of F by $\text{Iso}(F)$. It is not difficult to show that (2) is invariant under isometric motions of F [1], [15], [24].

Two frameworks (G, p) and $(G, \hat{p})$ are equivalent if $\phi(p) = \phi(\hat{p})$ and are congruent if $\|p_i - p_j\| = \|\hat{p}_i - \hat{p}_j\|$ for all $i, j \in V$ [13]. We say a framework (G, p) where $n > 2$ and p is generic[1] is infinitesimally rigid if and only if $\text{rank}[R(p)] = 2n - 3$ [8], [13]. A framework (G, p) is minimally rigid if $l = 2n - 3$. If two infinitesimally rigid frameworks (G, p) and $(G, \hat{p})$ are equivalent but not congruent, then they are said to be *flip ambiguous* [1]. We denote the set of all flip ambiguities of an infinitesimally rigid framework F and its isometries by $\text{Amb}(F)$. We assume that all frameworks in $\text{Amb}(F)$ are also infinitesimally rigid. This assumption is reasonable and, in fact, holds almost everywhere; see [1] and Theorem 3 of [3] for details.

The preliminary results below will be vital for establishing our main result. Specifically, they will allow us to formalize the stability set of the closed-loop system in relation to the infinitesimal rigidity and flip ambiguities of the framework modeling the formation. To this end, we consider two frameworks $F = (G, p)$ and $\bar{F} = (G, \bar{p})$ sharing the same graph $G = (V, E)$, and the metric [14]

$$\text{dist}(\zeta, \mathcal{M}) = \inf_{x \in M} \|\zeta - x\| \tag{5}$$

for a point ζ and a set $\mathcal{M}$.

Lemma 1: If F is infinitesimally rigid and $\text{dist}(\bar{p}, \text{Iso}(F)) \leq \varepsilon$ where ε is a sufficiently small positive constant, then $\bar{F}$ is also infinitesimally rigid.

[1]By generic, we mean the affine span of p is all of $\mathbb{R}^2$ [8].

Proof: Let $\hat{F} = (G, \hat{p}) \in \text{Iso}(F)$ be such that

$$\text{dist}(\bar{p}, \text{Iso}(F)) = \inf_{x \in \text{Iso}(F)} \|\bar{p} - x\| = \|\bar{p} - \hat{p}\|. \quad (6)$$

Since F is infinitesimally rigid, then so is $\hat{F}$. Therefore, $\text{rank}[R(\hat{p})] = 2n-3$ and there exists a $(2n-3) \times (2n-3)$ submatrix of $R(\hat{p})$, $R_s(\hat{p})$, such that $\det[R_s(\hat{p})] \neq 0$. The submatrix $R_s(\hat{p})$ has nonzero elements of the form $(\hat{p}_i - \hat{p}_j)^T$, $(i,j) \in E$. Since $\text{dist}(\bar{p}, \text{Iso}(F)) = \|\bar{p} - \hat{p}\| \leq \varepsilon$, it is not difficult to show that $[\bar{p}_i]_k = [\hat{p}_i]_k + \gamma_{ik}$ where $[\cdot]_k$ denotes the kth component of the vector and γ_{ik} is a sufficiently small positive constant. Thus, the nonzero elements of $R_s(\bar{p})$ have the form $[\bar{p}_i]_k - [\bar{p}_j]_k = [\hat{p}_i]_k - [\hat{p}_j]_k + \gamma_{ik} - \gamma_{jk}$, which are continuously dependent on $\hat{p}$. Since the eigenvalues of a matrix depend continuously on its elements [10], and the determinant of a matrix is the product of its eigenvalues, it follows that the determinant continuously depends on the elements of the matrix. Thus, for sufficiently small γ_{ik}, we have that $\det[R_s(\bar{p})] \neq 0$ and $\text{rank}[R_s(\bar{p})] = \text{rank}[R_s(\hat{p})] = 2n-3$. Now, since $R_s(\bar{p})$ is a full rank submatrix of $R(\bar{p})$, we know that $\text{rank}[R(\bar{p})] = 2n-3$ and the framework $\bar{F}$ is infinitesimally rigid. ■

Corollary 1: Consider the function

$$\Psi(\bar{F}, F) = \sum_{(i,j) \in E} (\|\bar{p}_i - \bar{p}_j\| - \|p_i - p_j\|)^2. \quad (7)$$

If F is infinitesimally rigid and $\Psi(\bar{F}, F) \leq \delta$ where δ is a sufficiently small positive constant, then $\bar{F}$ is also infinitesimally rigid.

Proof: First, note that $\Psi(\bar{F}, F) = 0$ implies that $\bar{F} \in \text{Iso}(F)$ or $\bar{F} \in \text{Amb}(F)$. Therefore, $\Psi(\bar{F}, F) \leq \delta$ implies that there is a sufficiently small positive constant ε such that $\text{dist}(\bar{p}, \text{Iso}(F)) \leq \varepsilon$ or $\text{dist}(\bar{p}, \text{Amb}(F)) \leq \varepsilon$. From Lemma 1, we know that $\bar{F}$ is infinitesimally rigid if $\text{dist}(\bar{p}, \text{Iso}(F)) \leq \varepsilon$. Since the elements of $\text{Amb}(F)$ are infinitesimally rigid, the proof of Lemma 1 can be followed with $\text{Iso}(F)$ replaced by $\text{Amb}(F)$ to show that $\text{dist}(\bar{p}, \text{Amb}(F)) \leq \varepsilon$ implies $\bar{F}$ is infinitesimally rigid. ■

Lemma 2: Let $v \in \mathbb{R}^2$ and $\mathbf{1}_n$ be the $n \times 1$ vector of ones, then $R(p)(\mathbf{1}_n \otimes v) = 0$.

Proof: From (3), it is not difficult to see that each row of the rigidity matrix $R(p)$ takes the form

$$\left[0 \; \ldots \; 0, \; (p_i - p_j)^T, 0 \; \ldots \; 0, \; (p_j - p_i)^T, \; 0 \; \ldots \; 0\right]. \quad (8)$$

Thus, the dot product of each row of $R(p)$ and the $2n \times 1$ vector $\mathbf{1}_n \otimes v$ will be zero. ■

III. PROBLEM STATEMENT

Consider a system of n agents in the plane modeled by the double integrator [22], [23]

$$\dot{q}_i = v_i \quad (9a)$$

$$\dot{v}_i = u_i, \quad i = 1, \ldots, n \quad (9b)$$

where $q_i = (x_i, y_i) \in \mathbb{R}^2$ is the ith agent position with respect to (w.r.t.) an Earth-fixed coordinate frame, $v_i \in \mathbb{R}^2$ represents the ith agent velocity w.r.t. an Earth-fixed coordinate frame, and $u_i \in \mathbb{R}^2$ is the (acceleration-level) control input.

Let the *desired formation* for the agents be represented by an infinitesimally and minimally rigid framework $F^* = (G^*, q^*)$ where $G^* = (V^*, E^*)$, $\dim(V^*) = n$, $\dim(E^*) = l$, and $q^* = (q_1^*, \ldots, q_n^*)$. The *constant* desired distance between agents i and j is given by

$$d_{ij} = \left\|q_i^* - q_j^*\right\| > 0, \quad i, j \in V^*. \quad (10)$$

Consider that the actual formation of the agents is represented by the framework $F(t) = (G^*, q(t))$ where $q = (q_1, \ldots, q_n)$. Assume that the relative position of agent pairs in E^* and the velocity of each agent can be measured. Further, assume that at $t = 0$ the agents do not satisfy the desired inter-agent distance constraints, i.e., $\|q_i(0) - q_j(0)\| \neq d_{ij}$, $i, j \in V^*$.

In this paper, we deal with two types of control problems for the multi-agent system: formation maneuvering and target interception. The common, primary control objective for these two problems is to design $u_i = u_i(q_i - q_j, d_{ij}, v_i - v_j, v_i)$, $i = 1, \ldots, n$ and $j \in \mathcal{N}_i(E^*)$ where $\mathcal{N}_i(\cdot)$ was defined in (1) such that

$$F(t) \to \text{Iso}(F^*) \text{ as } t \to \infty. \quad (11)$$

Note that (11) is equivalent to

$$\|q_i(t) - q_j(t)\| \to d_{ij} \text{ as } t \to \infty, \quad i, j \in V^*. \quad (12)$$

In the formation maneuvering problem, the secondary objective is

$$v_i(t) - v_d(t) \to 0 \text{ as } t \to \infty, \; i = 1, ..., n \quad (13)$$

where $v_d \in \mathbb{R}^2$ is the desired translational velocity for the swarm. We assume v_d is a known $\mathcal{C}^1$ function and $v_d(t), \dot{v}_d(t) \in \mathcal{L}_\infty$. This assumption is not restrictive because v_d is known a priori and can be pre-stored on each agent's onboard computer.

In the target interception problem, we will take the nth agent to be the leader while the remaining agents are followers. Our control scheme will consist of: a) selecting the infinitesimally and minimally rigid framework F^* such that $q_n^* \in \text{conv}\{q_1^*, q_2^*, ..., q_{n-1}^*\}$ where $\text{conv}\{\cdot\}$ denotes the convex hull, b) the leader intercepting the target, and c) the followers tracking the leader while maintaining the desired formation. Thus, the secondary objective for this problem is

$$q_T(t) \in \text{conv}\{q_1(t), q_2(t), ..., q_{n-1}(t)\} \text{ as } t \to \infty \quad (14)$$

where $q_T \in \mathbb{R}^2$ denotes the target position. We consider that q_T is a $\mathcal{C}^2$ function and $q_T(t), \dot{q}_T(t), \ddot{q}_T(t) \in \mathcal{L}_\infty$. Here, we will assume the signals $q_T - q_n$, $\dot{q}_T - \dot{q}_n$, and $\dot{q}_T$ are known and can be broadcast to the followers; however, $\ddot{q}_T$ is *unknown*. Knowledge of these signals is reasonable since they can be directly measured by sensors onboard the leader.

IV. FORMATION MANEUVERING

Define the relative position of two agents as

$$\tilde{q}_{ij} = q_i - q_j, \quad (i,j) \in E^*, \tag{15}$$

and let $\tilde{q} = (\ldots, \tilde{q}_{ij}, \ldots) \in \mathbb{R}^{2l}$ with the same ordering of terms as the edge function (2). The distance error is given by

$$e_{ij} = \|\tilde{q}_{ij}\| - d_{ij}, \quad (i,j) \in E^*. \tag{16}$$

It follows from (16) and (9a) that the distance error dynamics is

$$\dot{e}_{ij} = \frac{d}{dt}\left(\sqrt{\tilde{q}_{ij}^T \tilde{q}_{ij}}\right) = \frac{\tilde{q}_{ij}^T (v_i - v_j)}{e_{ij} + d_{ij}}. \tag{17}$$

Consider the potential function [9], [15], [18]

$$W_{ij} = \frac{1}{4} z_{ij}^2. \tag{18}$$

where

$$z_{ij} = \|\tilde{q}_{ij}\|^2 - d_{ij}^2, \quad (i,j) \in E^*. \tag{19}$$

Note that (19) can be rewritten using (16) as

$$z_{ij} = e_{ij}\left(\|\tilde{q}_{ij}\| + d_{ij}\right) = e_{ij}\left(e_{ij} + 2d_{ij}\right). \tag{20}$$

Since $\|\tilde{q}_{ij}\| \neq -d_{ij}$ because $\|\tilde{q}_{ij}\| \geq 0$ (or equivalently, $e_{ij} \neq -2d_{ij}$ because $e_{ij} \geq -d_{ij}$), it is easy to that $z_{ij} = 0$ if and only if $e_{ij} = 0$. Therefore, (18) is positive definite and radially unbounded in e_{ij}.

We now define the following function

$$W(e) = \sum_{(i,j)\in E^*} W_{ij}(e_{ij}) \tag{21}$$

where $e = (\ldots, e_{ij}, \ldots) \in \mathbb{R}^{l^*}$ is ordered as (2). The time derivative of (21) along (17) is given by

$$\dot{W} = \sum_{(i,j)\in E^*} e_{ij}\left(e_{ij} + 2d_{ij}\right)\tilde{q}_{ij}^T (v_i - v_j). \tag{22}$$

It follows from (3) and (20) that (22) can be rewritten as

$$\dot{W} = z^T R(q) v \tag{23}$$

where $v = (v_1, ..., v_n) \in \mathbb{R}^{2n}$, $z = (..., z_{ij}, ...) \in \mathbb{R}^{l^*}$, $(i,j) \in E^*$. The terms in z are ordered in the same way as in (2).

Following the backstepping technique [16], we introduce the variable

$$s = v - v_f \tag{24}$$

where $v_f \in \mathbb{R}^{2n}$ denotes the fictitious velocity input. We also introduce the function

$$W_d(e,s) = W(e) + \frac{1}{2} s^T s \tag{25}$$

where W was defined in (21). After taking the time derivative of (25), we obtain

$$\dot{W}_d = z^T R(q) v_f + s^T \left[u + R^T(q) z - \dot{v}_f\right] \tag{26}$$

where (23), (9b), and (24) were used, and $u = (u_1, ..., u_n) \in \mathbb{R}^{2n}$.

Before stating the main result, we introduce the following lemma and conjecture.

Lemma 3: For nonnegative constants c and δ, the level set $W(e) \leq c$ is equivalent to $\Psi(F, F^*) \leq \delta$ where Ψ and W were defined in (7) and (21), respectively.

Proof: First, from (7), (10), and (16), we have that

$$\begin{aligned} \Psi(F, F^*) &= \sum_{(i,j)\in E^*} \left(\|q_i - q_j\| - \left\|q_i^* - q_j^*\right\|\right)^2 \\ &= \sum_{(i,j)\in E^*} \left(\|q_i - q_j\| - d_{ij}\right)^2 \\ &= \sum_{(i,j)\in E^*} e_{ij}^2. \end{aligned} \tag{27}$$

From (21) and the conditions on $W_{ij}(e_{ij})$, we know $W(e) \leq c$ implies that e_{ij} is bounded for $(i,j) \in E^*$. This boundedness along with (27) implies $\Psi(F, F^*) \leq \delta$ where δ is some nonnegative constant.

Now, given $\Psi(F, F^*) \leq \delta$, it follows from (27) that e_{ij} is bounded for $(i,j) \in E^*$. This implies $W_{ij}(e_{ij})$ is bounded and $W(e) = \sum_{(i,j)\in E^*} W_{ij}(e_{ij}) \leq c$ where c is some nonnegative constant.

The control law for formation maneuvering is given in the following theorem. ■

Theorem 1: Given the formation $F(t) = (G^*, q(t))$, let the initial conditions be such that $(e(0), s(0)) \in \Omega \times \mathbb{R}^{2n}$ where

$$\begin{aligned} \Omega = \; &\{e \in \mathbb{R}^l \mid \Psi(F, F^*) \leq \delta \wedge \\ &\text{dist}(q, \text{Iso}(F^*)) < \text{dist}(q, \text{Amb}(F^*))\} \end{aligned} \tag{28}$$

and δ is a sufficiently small positive constant. Then, the control

$$u = -k_a s + \dot{v}_f - R^T(q) z, \tag{29}$$

where

$$v_f = u_a + (\mathbf{1}_n \otimes v_d), \tag{30}$$

$$u_a = -k_v R^T(q) z, \tag{31}$$

and $k_a, k_v > 0$, renders $(e,s) = 0$ exponentially stable and ensures that (11) and (13) are satisfied.

Proof: First, since F^* and $F(t)$ have necessarily the same number of edges, the minimal rigidity of F^* implies that $F(t)$ is minimally rigid for all $t \geq 0$. Substituting (29) into (26) and applying Lemma 2 yields

$$\dot{W}_d = -k_v z^T R(q) R^T(q) z - k_a s^T s. \tag{32}$$

Since F^* is infinitesimally rigid, we know from Corollary 1 that F is infinitesimally rigid for $e \in \Omega$. Since F is infinitesimally and minimally rigid for $e \in \Omega$, then $R(q)$ has full row rank. Since $\text{rank}[R(q)] = \text{rank}\left[R(q) R^T(q)\right]$, the matrix $R(q) R^T(q)$ is invertible for $e \in \Omega$; therefore,

$$\begin{aligned} \dot{W}_d &\leq -k_v \lambda_{\min}\left(R(q) R^T(q)\right) z^T z - k_a s^T s \\ &\leq -\min\left\{4k_v \lambda_{\min}\left(RR^T\right), 2k_a\right\} W_d \end{aligned} \tag{33}$$

for $(e(t), s(t)) \in \Omega \times \mathbb{R}^{2n}$, where (25) was used and $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue. From Lemma 3 and the negative definiteness of (33), we know the level surfaces of W_d are invariant [14] and, if $(e(0), s(0)) \in \Omega \times \mathbb{R}^{2n}$, then $(e(t), s(t))$ stays in $\Omega \times \mathbb{R}^{2n}$ for all $t > 0$. Thus, from the form of (33), $(e, s) = 0$ is exponentially stable for $(e(0), s(0)) \in \Omega \times \mathbb{R}^{2n}$ [14]. The exponential stability of $e = 0$ implies that $F(t) \to \mathrm{Iso}(F^*)$ or $F(t) \to \mathrm{Amb}(F^*)$ as $t \to \infty$. However, since $e(0) \in \Omega$, we have from (28) that

$$\mathrm{dist}(q(0), \mathrm{Iso}(F^*(0))) < \mathrm{dist}(q(0), \mathrm{Amb}(F^*(0))). \quad (34)$$

We know from (33) that $W_d(t)$ is decreasing or constant for all $t \geq 0$. Due to (34), the energy-like function $W(t)$ defined in (21) would need to increase for a period of time for $F(t) \to \mathrm{Amb}(F^*)$ as $t \to \infty$, which is a contradiction. Therefore, we know $F(t) \to \mathrm{Iso}(F^*)$ as $t \to \infty$ for $e(0) \in \Omega$.

Finally, since $e(t) \to 0$ as $t \to \infty$ from the above analysis, we know from (20) that $z(t) \to 0$ as $t \to \infty$. Since e is bounded, we know from (16) that $\tilde{q}$ is bounded. Therefore, $R(q)$ is bounded and we have that $u_a(t) \to 0$ as $t \to \infty$ from (31). From (30), we then know that $v_f(t) \to (\mathbf{1}_n \otimes v_d)$ as $t \to \infty$. Since we have proven that $s(t) \to 0$ as $t \to \infty$, it follows from (24) that $v(t) - v_f(t) \to 0$ as $t \to \infty$. Therefore, $v_i(t) - v_d(t) \to 0$ as $t \to \infty$, $i = 1, ..., n$. ■

Remark 1: The condition $e(0) \in \Omega$ in Theorem 1 implies that the actual formation $F(t)$ needs to be sufficiently close to a framework in $\mathrm{Iso}(F^*)$ at $t = 0$ to avoid a flip ambiguity while remaining infinitesimally rigid. Therefore, the set in (28) exists because it is always possible to select $F(0)$ in this manner. We note that the local nature of the stability result in the Cartesian plane is common in the formation control literature based on rigid graph theory as it is inherent to the approach; see, e.g., [15], [18], [26]. In practice, the region of attraction of the control is not necessarily small as can be seen from simulations for different initial conditions.

Remark 2: The expression for $\dot{v}_f$ in (29) is given by

$$\dot{v}_f = -k_v \dot{R}^T z - k_v R^T \dot{z} + (\mathbf{1}_n \otimes \dot{v}_d) \quad (35)$$

where from (3)

$$\dot{R} = R(v) \quad (36)$$

and from (19) and (23)

$$\dot{z} = 2R(q)v. \quad (37)$$

Remark 3: The control (29) can be written componentwise as follows

$$\begin{aligned} u_i \; = \; & \dot{v}_d - k_a v_i - \sum_{j \in \mathcal{N}_i(E^*)} [(k_a k_v + 1)\, \tilde{q}_{ij} z_{ij} \\ & + k_v \left(z_{ij} I_2 + 2\tilde{q}_{ij}\tilde{q}_{ij}^T\right) \tilde{v}_{ij}] \end{aligned} \quad (38)$$

where I_2 is the 2×2 identity matrix and $\tilde{v}_{ij} = v_i - v_j$, $(i, j) \in E^*$. Thus, the control for each agent does not require measurement of the absolute position of the agents. Rather, it only requires that the ith agent measure its own velocity and the relative position/velocity with respect to its neighbors $\mathcal{N}_i(E^*)$. Note that the minimal rigidity of the graph helps reduce the size of $\mathcal{N}_i(E^*)$; thus, facilitating the control implementation for large-scale systems. The control is also dependent on v_d and $\dot{v}_d$, but since these signals are known a priori they can be stored on each agent's onboard computer.

V. TARGET INTERCEPTION

We now turn our attention to the target interception problem. To this end, we define the target's relative position to the leader (i.e., target interception error) as

$$e_T := q_T - q_n \quad (39)$$

and let $v_T := \dot{q}_T$. Also, let $\|x\|_{\mathcal{L}_\infty} := \sup_{t \geq 0} \|x(t)\|$ for any piecewise continuous, bounded function $x : [0, \infty] \to \mathbb{R}^n$ [14]. The theorem below gives the main result of this section.

Theorem 2: Let the initial conditions be such that $(q(0), s(0)) \in \Omega \times \mathbb{R}^{2n}$ where Ω was defined in Theorem 1. Consider the control

$$u = -k_a s + h - R^T(q)\, z \quad (40)$$

where s was defined in (24),

$$v_f = u_a + \mathbf{1}_n \otimes (v_T + k_T e_T), \quad (41)$$

$$h = \dot{u}_a - k_s \mathrm{sgn}(s) + \mathbf{1}_n \otimes k_T v_T, \quad (42)$$

u_a was defined in (31), $\mathrm{sgn}(x) = (sgn(x_1), \ldots, sgn(x_m))$, $\forall x \in \mathbb{R}^m$, $sgn(\cdot)$ is the standard signum function, $k_a, k_T > 0$, and $k_s \geq \sqrt{n}\, \|\dot{v}_T\|_{\mathcal{L}_\infty}$. Then, (40) renders $(e, s) = 0$ asymptotically stable and ensures that (11) and (14) are satisfied.

Proof: First, notice that the differential equations describing the error system in closed loop with (40)-(42) have a discontinuous right-hand side. That is, if $\dot{\xi} = f(\xi, t)$ denotes the closed-loop system where $\xi = (e, s)$, then $f(\xi, t)$ is continuous everywhere except in the set $\{(\xi, t) \mid s = 0\}$. For such a system, one can show the existence of generalized solutions by embedding the differential equations into the differential inclusions $\dot{\xi} \in K[f](\xi, t)$ where $K[\cdot]$ is a nonempty, compact, convex, upper semicontinuous set-valued map [25]. In this case, the time derivative of (25) is given by [25]

$$\begin{aligned} \dot{W}_d \; & \overset{a.e.}{\in} \frac{\partial W_d}{\partial \xi} K[f](\xi, t) \\ & \subset z^T R(q) v_f + s^T \left[u + R^T(q)\, z - \dot{v}_f\right] \end{aligned} \quad (43)$$

where (26) was used.

Substituting (40) into (43) along with (31), (41), and (42), and applying Lemma 2 gives [25]

$$\begin{aligned} \dot{W}_d \; & \subset \; -k_v z^T R R^T z - k_a s^T s - s^T [k_s \mathrm{sgn}(s) \\ & \qquad + \mathbf{1}_n \otimes \dot{v}_T] \\ & = \; -k_v z^T R R^T z - k_a s^T s - s^T [k_s \mathrm{SGN}(s) \\ & \qquad + \mathbf{1}_n \otimes \dot{v}_T] \\ & \leq \; -k_v z^T R R^T z - k_a s^T s \\ & \qquad + \|s\| \left(\sqrt{n}\, \|\dot{v}_T\|_{\mathcal{L}_\infty} - k_s\right) \end{aligned} \quad (44)$$

where $\mathrm{SGN}(x) = (SGN(x_1), \ldots, SGN(x_m))$, $\forall x \in \mathbb{R}^m$ and

$$SGN(x_i) = \begin{cases} 1 & \text{for } x_i > 0 \\ [-1, 1] & \text{for } x_i = 0 \\ -1 & \text{for } x_i < 0. \end{cases} \quad (45)$$

Given that $k_s \geq \sqrt{n}\left\|\dot{v}_T\right\|_{\mathcal{L}_\infty}$, we can follow the proof of Theorem 1 to show that (44) is negative definite for $(q(0), s(0)) \in \Omega \times \mathbb{R}^{2n}$. Therefore, from Theorem 3.1 in [25], we know that $(e, s) = 0$ is asymptotically stable. The proof that (11) holds follows from the same arguments used in the proof of Theorem 1.

Now, note from (41), that

$$v_{fn} = u_{an} + v_T + k_T e_T \tag{46}$$

where the subscript n denotes the nth element of the corresponding vector. Differentiating (39) and applying (46) yields

$$\dot{e}_T = v_T - v_n = v_T - (v_{fn} + s_n) = -k_1 e_T + r \tag{47}$$

where $r := -s_n - u_{an}$. Note that (47) is a stable linear system with input r. Since $(e, s) = 0$ is asymptotically stable, we can show as in the proof of Theorem 1 that $u_a(t) \to 0$ as $t \to \infty$. Therefore, $r(t) \to 0$ as $t \to \infty$ and, from (47), $e_T(t) \to 0$ as $t \to \infty$.

Finally, since we know the control (40) ensures $q_n(t) \in \text{conv}\{q_1(t), q_2(t), ..., q_{n-1}(t)\}$ as $t \to \infty$ due to the manner in which F^* is constructed, we conclude from the convergence of (39) to zero that (14) holds. ■

Remark 4: The control (40) can be written componentwise as follows

$$\begin{aligned} u_i \;=\; & -k_a v_i + k_a\left(v_T + k_T e_T\right) + k_T v_T \\ & - \sum_{j \in \mathcal{N}_i}\left[k_v\left(z_{ij} I_2 + 2\tilde{q}_{ij}\tilde{q}_{ij}^T\right)\tilde{v}_{ij}\right. \\ & \left. + \left(k_a k_v + 1\right)\tilde{q}_{ij} z_{ij}\right] \\ & -k_s \operatorname{sgn}(v_i - v_T - k_T e_T + k_v \sum_{j \in \mathcal{N}_i}\tilde{q}_{ij} z_{ij}) \end{aligned} \tag{48}$$

where the argument in $\mathcal{N}_i(E^*)$ was omitted for simplicity. Therefore, a conclusion similar to the one in Remark 3 can be drawn. The difference here is that the control input for each agent is also dependent on e_T and v_T.

VI. CONCLUSIONS

We constructed new control laws for stabilizing inter-agent distances to pre-defined values while allowing the formation to follow a time-varying trajectory or intercept and surround a moving target on the plane. A leader-follower approach was used for solving the target interception problem. The proposed controllers are composed of a formation acquisition term and a formation maneuvering or target interception term. In both problems, we measure the relative position and velocity of agents connected in the infinitesimally and minimally rigid graph along with the agent's own absolute velocity. For formation maneuvering, the desired trajectory velocity is available to all agents. In the target interception problem, we also measure the relative position of the target to the leader and the target absolute velocity. This information is broadcast by the leader to all followers. The target acceleration is assumed to be unknown but bounded.

REFERENCES

[1] B.D.O. Anderson, C. Yu, B. Fidan, and J.M. Hendrickx, "Rigid graph control architectures for autonomous formations," *IEEE Contr. Syst. Mag.*, vol. 28, no. 6, pp. 48-63, 2008.

[2] L. Asimow and B. Roth, "The rigidity of graphs II," *J. Math. Anal. Appl.,* vol. 68, no. 1, pp. 171-190, 1979.

[3] J. Aspnes, J. Egen, D.K. Goldenberg, A.S. Morse, W. Whiteley, Y.R. Yang, B.D.O. Anderson, and P.N. Belhumeur, "A theory of network localization," *IEEE Trans. Mob. Comput.*, vol. 5, no. 12, 1663-1678, 2006.

[4] X. Cai and M. de Queiroz, "On the stabilization of planar multi-agent formations," *Proc. ASME Conf. Dyn. Syst. Contr.*, Paper No. DSCC2012-MOVIC2012-8534, Ft. Lauderdale, FL, 2012.

[5] X. Cai and M. de Queiroz, "Multi-agent formation maintenance and target tracking," *Proc. Amer. Contr. Conf.*, pp. 2537-2532, Washington, DC, 2013.

[6] X. Cai and M. de Queiroz, "Rigidity-based stabilization of multi-agent formations," *ASME J. Dyn. Syst. Measur. Contr.*, vol. 136, no. 1, Paper 014502, 2014.

[7] M. Cao, A.S. Morse, C. Yu, B.D.O. Anderson, and S. Dasgupta, "Maintaining a directed, triangular formation of mobile autonomous agents," *Commun. Inf. Syst.*, vol. 11, no. 1, pp. 1-16, 2011.

[8] R. Connelly, "Generic global rigidity," *Discrete Comput. Geom.*, vol. 33, no. 4, pp. 549-563, 2005.

[9] F. Dörfler and B. Francis, "Geometric analysis of the formation problem for autonomous robots," *IEEE Trans. Autom. Contr.*, vol. 55, no. 10, pp. 2379-2384, 2010.

[10] J.N. Franklin, *Matrix theory*, Englewood Cliffs, New Jersey: Prentice Hall, 1968.

[11] V. Gazi and K.M. Passino, *Swarm stability and optimization*, Berlin: Springer, 2011.

[12] I. Izmestiev, "Infinitesimal rigidity of frameworks and surfaces," *Lectures on Infinitesimal Rigidity*, Kyushu University, Japan, 2009.

[13] B. Jackson, "Notes on the rigidity of graphs," *Notes of the Levico Conference*, 2007.

[14] H.K. Khalil, *Nonlinear systems*, Englewood Cliffs, New Jersey: Prentice Hall, 2002.

[15] L. Krick, M.E. Broucke and B.A. Francis, "Stabilization of infinitesimally rigid formations of multi-robot networks," *Intl. J. Contr.*, vol. 83, no. 3, pp. 423-439, 2009.

[16] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and adaptive control design*, New York, NY: John Wiley & Sons, 1995.

[17] D. Meng, Y. Jia, J. Du, and F. Yu, "Tracking control over a finite interval for multi-agent systems with a time-varying reference trajectory," *Syst. Contr. Lett.*, vol. 61, no. 7, pp. 807-818, 2012.

[18] K.-K. Oh, and H.-S. Ahn, "Formation control of mobile agents based on inter-agent distance dynamics," *Automatica*, vol. 47, no. 10, pp. 2306-2312, 2011.

[19] K.-K. Oh, and H.-S. Ahn, "Distance-based control of cycle-free persistent formations," *Proc. IEEE Multi-Conf. Systems and Control*, pp. 816-821, Denver, CO, 2011.

[20] K.-K. Oh and H.-S. Ahn, "Distance-based undirected formations of single-integrator and double-integrator modeled agents in *n*-dimensional space," *Intl. J. Rob. Nonl. Contr.*, DOI: 10.1002/rnc.2967, 2013.

[21] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: algorithms and theory," *IEEE Trans. Autom. Contr.*, vol 51, no. 3, pp. 401-420, 2006.

[22] R. Olfati-Saber and P. Jalalkamali, "Coupled distributed estimation and control for mobile sensor networks," *IEEE Trans. Autom. Contr.*, vol. 57, no. 10, pp. 2609-2614, 2012.

[23] W. Ren and R.W. Beard, *Distributed consensus in multi-vehicle cooperative control*, London: Springer-Verlag, 2008.

[24] B. Roth, "Rigid and flexible frameworks," *The Amer. Math. Monthly,* vol. 86, no. 1, pp. 6-21, 1981.

[25] D. Shevitz and B. Paden, "Lyapunov stability of nonsmooth systems," *IEEE Trans. Autom. Contr.*, vol. 39, no. 9, pp. 1910-1914, 1994.

[26] T.H. Summers, C. Yu, S. Dasgupta, and B.D.O. Anderson, "Control of minimally persistent leader-remote-follower and coleader formations in the plane," *IEEE Trans. Autom. Contr,*, vol. 56, no. 12, pp. 2778-2792, 2011.

[27] F. Xiao, L. Wang, J. Chen, and Y. Gao, "Finite-time formation control for multi-agent systems," *Automatica*, vol. 45, no. 11, pp. 2605-2611, 2009.